Frequency-Domain Analysis of Transmission Line Circuits

(Part 1)

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1

Outline

- *N*-port networks
- Impedance matrix representation
- Admittance matrix representation
- Scattering matrix representation
- Meaning of the S-parameters
- Generalized S-parameters

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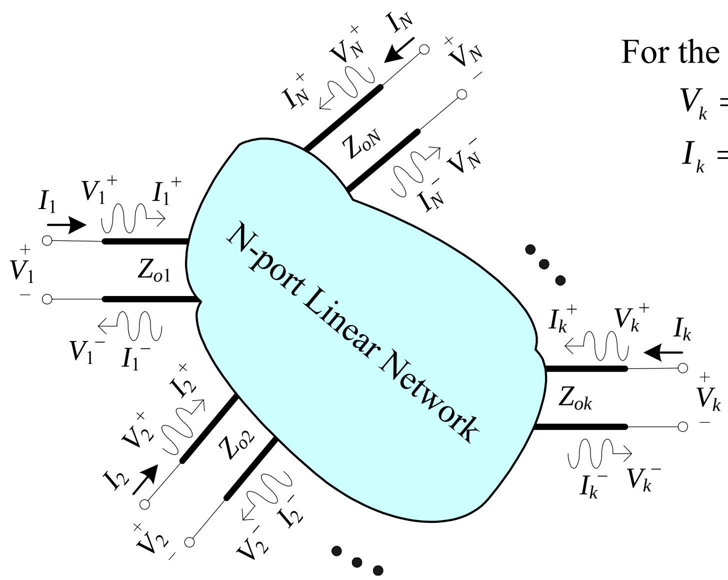
2

An N-port High-Speed Network

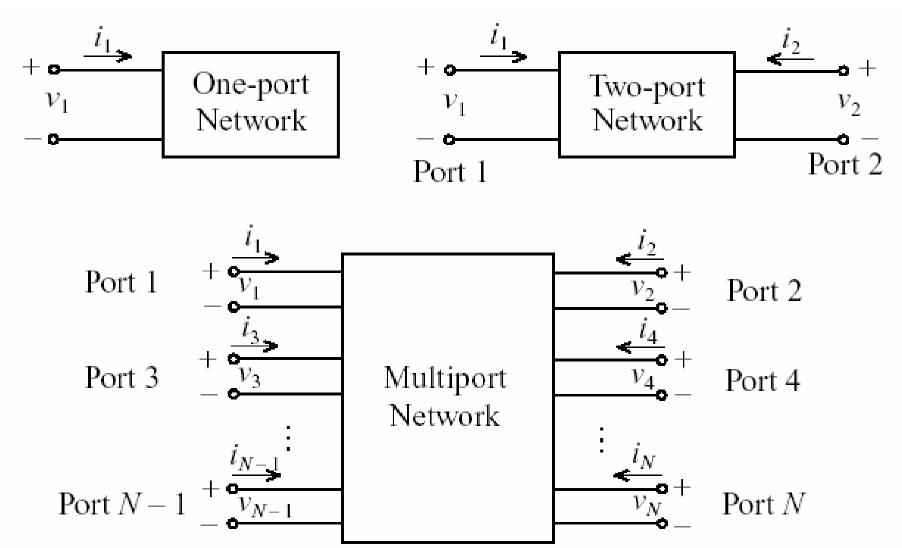


For the *k*-th port:

$$V_k = V_k^+ + V_k^-$$

$$I_k = I_k^+ + I_k^-$$

Examples of N-port Networks



(R. Ludwig and P. Bretchko, RF Circuit Design, Prentice Hall, 2000)

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Impedance Matrix Representation ($\boldsymbol{Z}$)

$$\boldsymbol{V} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix} \qquad \boldsymbol{I} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}$$

$$\boldsymbol{V} = \boldsymbol{Z}\boldsymbol{I}$$

$$\boldsymbol{Z} = \begin{bmatrix} Z_{11} & Z_{12} & \dots & Z_{1N} \\ Z_{21} & Z_{22} & \dots & Z_{2N} \\ \vdots & & & \\ Z_{N1} & Z_{N2} & \dots & Z_{NN} \end{bmatrix}$$

Each element of matrix $\boldsymbol{Z}$ is given by

$$Z_{ij} = \left. \frac{V_i}{I_j} \right|_{I_k = 0 \text{ for } k \neq j}$$

Admittance Matrix Representation ($\boldsymbol{Y}$)

$$\boldsymbol{V} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix} \qquad \boldsymbol{I} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}$$

$$\boldsymbol{I} = \boldsymbol{Y}\boldsymbol{V}$$

$$\boldsymbol{Y} = \begin{bmatrix} Y_{11} & Y_{12} & \dots & Y_{1N} \\ Y_{21} & Y_{22} & \dots & Y_{2N} \\ \vdots & & & \\ Y_{N1} & Y_{N2} & \dots & Y_{NN} \end{bmatrix}$$

Each element of matrix $\boldsymbol{Y}$ is given by

$$Y_{ij} = \left. \frac{I_i}{V_j} \right|_{V_k = 0 \text{ for } k \neq j}$$

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Some Properties of the *Z* an *Y* Matrices

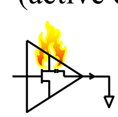
- If the network is reciprocal (not containing any nonreciprocal media such as ferrites or plasmas), then ***Z*** and ***Y*** are symmetric,

$$Z_{ij} = Z_{ji} \qquad Y_{ij} = Y_{ji}$$

- If the network is lossless, all the elements in ***Z*** and ***Y*** are purely imaginary
- In most practical cases, high-speed interconnects can be considered as reciprocal networks, and in many cases, lossless networks

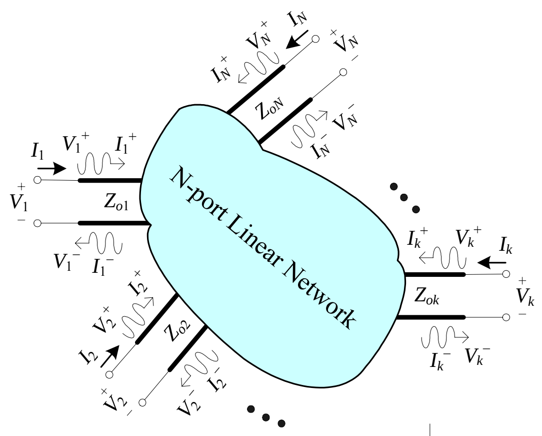
The Scattering Matrix Representation (*S*)

- They can be more easily obtained at high frequencies:
  - Incident and reflected waves can be measured using a Vector Network Analyzer (VNA)
  - They do not require "shorts" or "opens" (active devices might oscillate or self-destroy)



- They are more directly related to high-frequency effects (Γ, T, IL, RL, SWR, etc.)
- We can convert back and forth between S, Y and Z parameters

The Scattering Matrix ($\boldsymbol{S}$)



$$\boldsymbol{V}^+ = \begin{bmatrix} V_1^+ \\ V_2^+ \\ \vdots \\ V_N^+ \end{bmatrix} \qquad \boldsymbol{V}^- = \begin{bmatrix} V_1^- \\ V_2^- \\ \vdots \\ V_N^- \end{bmatrix}$$

$$\boldsymbol{V}^- = \boldsymbol{S}\boldsymbol{V}^+$$

$$\boldsymbol{S} = \begin{bmatrix} S_{11} & S_{12} & \dots & S_{1N} \\ S_{21} & S_{22} & \dots & S_{2N} \\ \vdots & & & \\ S_{N1} & S_{N2} & \dots & S_{NN} \end{bmatrix}$$

$$S_{ij} = \left. \frac{V_i^-}{V_j^+} \right|_{V_k^+ = 0 \text{ for } k \neq j}$$

$V_k^+ = 0$ if we terminate port k with a matched load ($Z_{LK} = Z_{ok}$)

Meaning of the S-parameters

- S_{ii} represents the reflection coefficient at port i when all the remaining ports are terminated with matched loads
- S_{ij} represents the transmission coefficient from port j to port i when all the other ports are terminated with matched loads

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Meaning of the S-parameters (cont)

S11 = forward reflection coefficient ***(input match)***
S22 = reverse reflection coefficient ***(output match)***
S21 = forward transmission coefficient ***(gain or loss)***
S12 = reverse transmission coefficient ***(isolation)***

$$S_{ii} \neq \Gamma_i \qquad S_{ii} = \left.\frac{V_i^-}{V_i^+}\right|_{V_k^+=0 \text{ for } k\neq i} = \left.\Gamma_i\right|_{V_k^+=0 \text{ for } k\neq i}$$

$$S_{ij} \neq T_{ji} \qquad S_{ij} = \left.\frac{V_i^-}{V_j^+}\right|_{V_k^+=0 \text{ for } k\neq j} = \left.T_{ji}\right|_{V_k^+=0 \text{ for } k\neq j}$$

Conversion between *S* and *Y* and *Z* Matrices

$$\boldsymbol{S} = (\boldsymbol{Z}+\boldsymbol{1})^{-1}(\boldsymbol{Z}-\boldsymbol{1})$$

$$\boldsymbol{Z} = (\boldsymbol{1}-\boldsymbol{S})^{-1}(\boldsymbol{1}+\boldsymbol{S})$$

(normalized ***Z***, assuming all ports have the same Z_o)

where ***1*** is the identity matrix (*N* by *N*)

$$\boldsymbol{Z} = \boldsymbol{Y}^{-1}$$

$$\boldsymbol{Y} = \boldsymbol{Z}^{-1}$$

Conversion between *S* and *Y* and *Z* (cont)

$$\boldsymbol{S} = (\boldsymbol{Z}\boldsymbol{Y}_o + \boldsymbol{1})^{-1}(\boldsymbol{Z}\boldsymbol{Y}_o - \boldsymbol{1})$$

$$\boldsymbol{Z} = \boldsymbol{Y}_o^{-1}(\boldsymbol{1} - \boldsymbol{S})^{-1}(\boldsymbol{1} + \boldsymbol{S})$$

where ***1*** is the identity matrix (N by N) and

$$\boldsymbol{Y}_o = diag(1/Z_{o1} \quad 1/Z_{o2} \quad \ldots \quad 1/Z_{oN})$$

$$\boldsymbol{Z} = \boldsymbol{Y}^{-1}$$

$$\boldsymbol{Y} = \boldsymbol{Z}^{-1}$$

Some Properties of the *S* Matrix

- If the network is reciprocal (not containing any nonreciprocal media such as ferrites or plasmas), then ***S*** is symmetric,

$$S_{ij} = S_{ji}$$

- If the network is lossless, matrix ***S*** is unitary (orthonormal complex)

$$\sum_{k=1}^{N} S_{ki} S_{ki}^{*} = 1 \qquad \sum_{k=1}^{N} S_{ki} S_{kj}^{*} = 0$$

- In most practical cases, high-speed interconnects can be considered as reciprocal networks, and in some cases, lossless networks

Generalized Scattering Parameters

- In many practical cases,

$$Z_{o1} = Z_{o2} = \ldots = Z_{oN} = Z_o = 50\Omega$$

- If Z_o is different for each port, we define the incident and reflected waves at the *k*-th port as

$$a_k = \frac{V_k^+}{\sqrt{Z_{ok}}} \qquad b_k = \frac{V_k^-}{\sqrt{Z_{ok}}}$$

- The total voltage and current at port *k*-th are

$$V_k = V_k^+ + V_k^- = (a_k + b_k)\sqrt{Z_{ok}}$$

$$I_k = I_k^+ + I_k^- = \frac{V_k^+}{Z_{ok}} - \frac{V_k^-}{Z_{ok}} = (a_k - b_k)/\sqrt{Z_{ok}}$$

Generalized Scattering Parameters (cont)

$$V_k = (a_k + b_k)\sqrt{Z_{ok}} \qquad I_k = (a_k - b_k)/\sqrt{Z_{ok}}$$

- The average power delivered to the *k*-th port is

$$P_k = \frac{1}{2}\text{Re}\{V_k I_k^*\}$$

$$P_k = \frac{1}{2}\text{Re}\{|a_k|^2 - |b_k|^2 + (b_k a_k^* - a_k b_k^*)\} = \frac{1}{2}|a_k|^2 - \frac{1}{2}|b_k|^2$$

(incident power minus reflected power)

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Generalized Scattering Parameters - Summary

- The Generalized Scattering Matrix relates the incident and reflected waves

$$\boldsymbol{b} = \boldsymbol{S}\boldsymbol{a}$$

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix} \quad S_{ij} = \left.\frac{b_i}{a_j}\right|_{V_k^+=0 \text{ for } k\neq j} \quad a_j = \frac{V_j^+}{\sqrt{Z_{oj}}} \quad b_i = \frac{V_i^-}{\sqrt{Z_{oi}}}$$

- To convert from the original S-parameters to generalized S-parameters

$$S_{ij} = \left.\frac{V_i^- \sqrt{Z_{oj}}}{V_j^+ \sqrt{Z_{oi}}}\right|_{V_k^+=0 \text{ for } k\neq j}$$

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