Frequency-Domain Analysis of Transmission Line Circuits
(Part 1)

Dr. José Ernesto Rayas-Sánchez

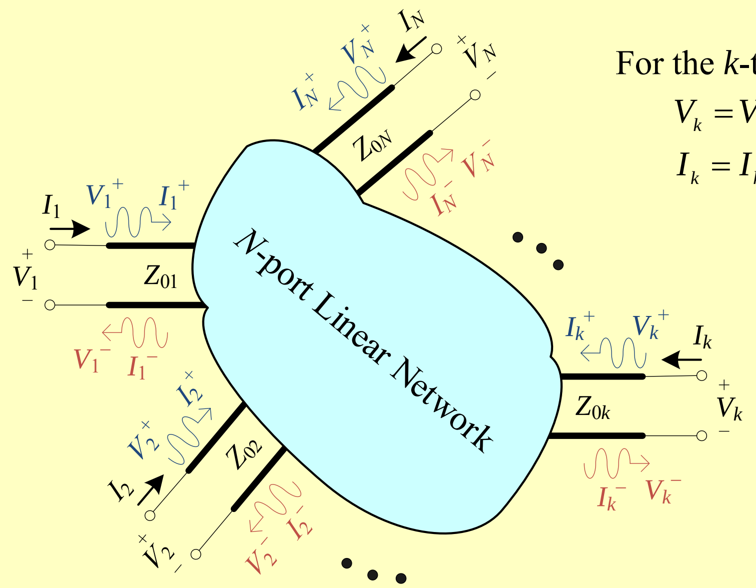
1

Outline

- *N*-port networks
- Impedance matrix representation (Z-parameters)
- Admittance matrix representation (Y-parameters)
- Hybrid matrix representation (H-parameters)
- Scattering matrix representation (S-parameters)
- Meaning of the S-parameters
- Conversion between Y-Z-S parameters
- Properties of the S matrix

Dr. J. E. Rayas-Sánchez 2

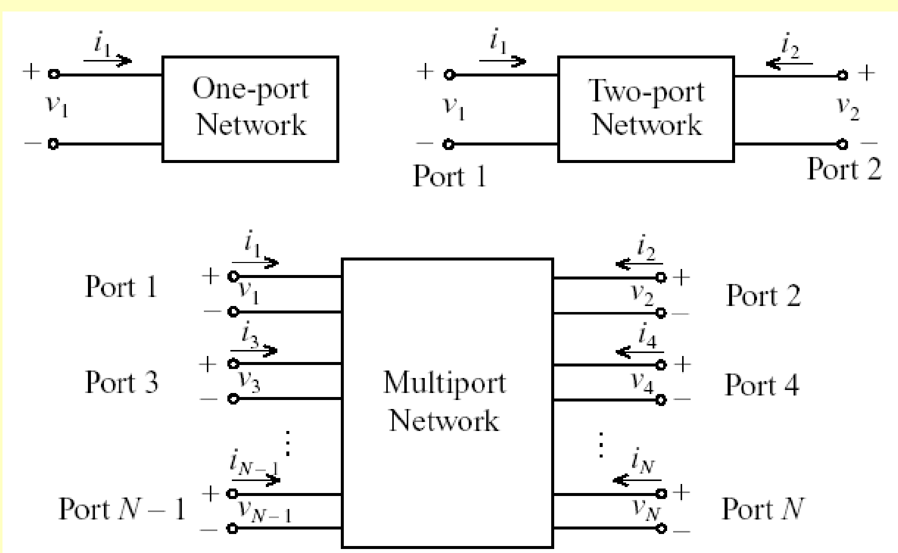
An *N*-port High-Speed Network



For the *k*-th port:

$$V_k = V_k^+ + V_k^-$$
$$I_k = I_k^+ + I_k^-$$

Examples of N-port Networks



(R. Ludwig and P. Bretchko, RF Circuit Design, Prentice Hall, 2000)

Impedance Matrix Representation ($\boldsymbol{Z}$)

$$\boldsymbol{V} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix} \qquad \boldsymbol{I} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}$$

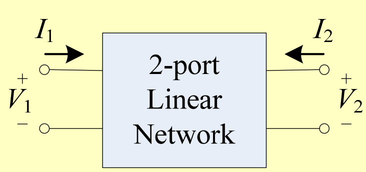
$$\boldsymbol{V} = \boldsymbol{Z}\boldsymbol{I}$$

$$\boldsymbol{Z} = \begin{bmatrix} Z_{11} & Z_{12} & \dots & Z_{1N} \\ Z_{21} & Z_{22} & \dots & Z_{2N} \\ \vdots & & & \\ Z_{N1} & Z_{N2} & \dots & Z_{NN} \end{bmatrix}$$

Each element of matrix $\boldsymbol{Z}$ is given by

$$Z_{ij} = \left. \frac{V_i}{I_j} \right|_{I_k = 0 \text{ for } k \neq j}$$

Z-Parameters for 2-Port Networks

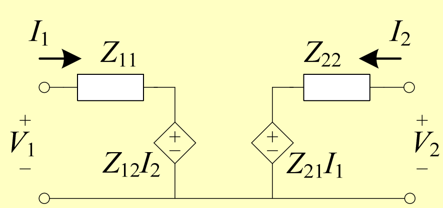


$$\boldsymbol{V} = \boldsymbol{Z}\boldsymbol{I}$$

$$\boldsymbol{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \qquad \boldsymbol{I} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$\boldsymbol{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

Equivalent circuit:



Admittance Matrix Representation (*Y*)

$$\boldsymbol{V} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix} \qquad \boldsymbol{I} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}$$

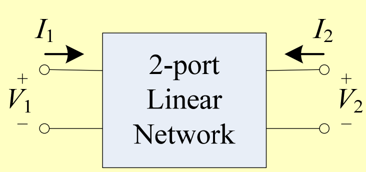
$$\boldsymbol{I} = \boldsymbol{YV}$$

$$\boldsymbol{Y} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1N} \\ Y_{21} & Y_{22} & \cdots & Y_{2N} \\ \vdots & & & \\ Y_{N1} & Y_{N2} & \cdots & Y_{NN} \end{bmatrix}$$

Each element of matrix ***Y*** is given by

$$Y_{ij} = \left.\frac{I_i}{V_j}\right|_{V_k=0 \text{ for } k \neq j}$$

Y-Parameters for 2-Port Networks

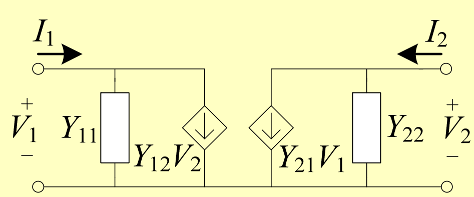


$$\boldsymbol{I} = \boldsymbol{YV}$$

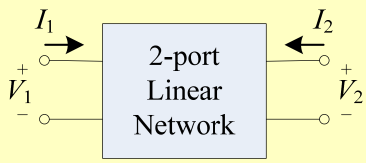
$$\boldsymbol{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \qquad \boldsymbol{I} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$\boldsymbol{Y} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$$

Equivalent circuit:



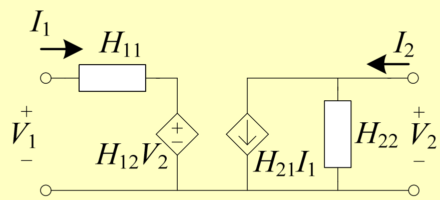
H-Parameters (Hybrid) for 2-Port Networks



$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \boldsymbol{H} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

$$\boldsymbol{H} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

Equivalent circuit:



Some Properties of the *Z* an *Y* Matrices

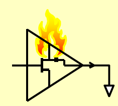
- If the network is reciprocal (passive and not containing any nonreciprocal media such as ferrites or plasmas), then ***Z*** and ***Y*** are symmetric,

$$Z_{ij} = Z_{ji} \qquad Y_{ij} = Y_{ji}$$

- If the network is lossless, all the elements in ***Z*** and ***Y*** are purely imaginary
- In most practical cases, high-speed interconnects can be considered as reciprocal networks, and in some cases, lossless networks

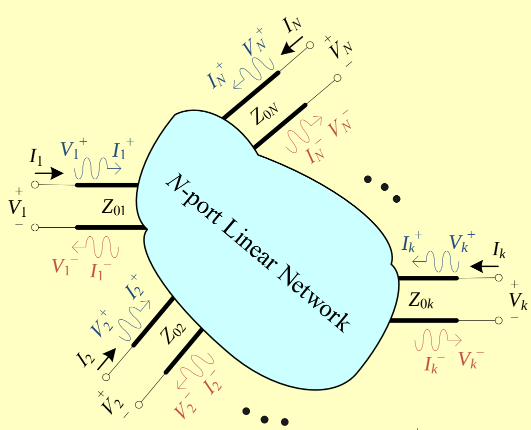
The Scattering Matrix Representation (*S*)

- They can be more easily obtained at high frequencies:
  - Incident and reflected waves can be measured using a Vector Network Analyzer (VNA)
  - They do not require "shorts" or "opens" (active devices might oscillate or self-destroy)



- They are more directly related to high-frequency effects (Γ, T, IL, RL, SWR, etc.)
- We can convert back and forth between S, Y and Z parameters

The Scattering Matrix (*S*)



$$\boldsymbol{V}^+ = \begin{bmatrix} V_1^+ \\ V_2^+ \\ \vdots \\ V_N^+ \end{bmatrix} \qquad \boldsymbol{V}^- = \begin{bmatrix} V_1^- \\ V_2^- \\ \vdots \\ V_N^- \end{bmatrix}$$

$$\boldsymbol{V}^- = \boldsymbol{S}\boldsymbol{V}^+$$

$$\boldsymbol{S} = \begin{bmatrix} S_{11} & S_{12} & \dots & S_{1N} \\ S_{21} & S_{22} & \dots & S_{2N} \\ \vdots & & & \\ S_{N1} & S_{N2} & \dots & S_{NN} \end{bmatrix}$$

$$S_{ij} = \left. \frac{V_i^-}{V_j^+} \right|_{V_k^+ = 0 \text{ for } k \neq j}$$

$V_k^+ = 0$ if we terminate port k with a matched load ($Z_{Lk} = Z_{0k}$)

Meaning of the S-parameters

- S_{ii} represents the reflection coefficient at port *i* when all the remaining ports are terminated with matched loads
- S_{ij} represents the transmission coefficient from port *j* to port *i* when all the other ports are terminated with matched loads

Meaning of the S-parameters (cont.)

S_{11} : forward reflection coefficient (input match)
S_{22} : reverse reflection coefficient (output match)
S_{21} : forward transmission coefficient (gain or loss)
S_{12} : reverse transmission coefficient (isolation)

$$S_{ii} \neq \Gamma_i \qquad S_{ii} = \left.\frac{V_i^-}{V_i^+}\right|_{V_k^+=0 \text{ for } k\neq i} = \left.\Gamma_i\right|_{V_k^+=0 \text{ for } k\neq i}$$

$$S_{ij} \neq T_{ji} \qquad S_{ij} = \left.\frac{V_i^-}{V_j^+}\right|_{V_k^+=0 \text{ for } k\neq j} = \left.T_{ji}\right|_{V_k^+=0 \text{ for } k\neq j}$$

Conversion between $\boldsymbol{S}$ and $\boldsymbol{Y}$ and $\boldsymbol{Z}$ Matrices

$$\left.\begin{aligned} \boldsymbol{S} &= (\boldsymbol{Z}+\boldsymbol{1})^{-1}(\boldsymbol{Z}-\boldsymbol{1}) \\ \boldsymbol{Z} &= (\boldsymbol{1}+\boldsymbol{S})(\boldsymbol{1}-\boldsymbol{S})^{-1} \end{aligned}\right\} \text{(normalized } \boldsymbol{Z}\text{, assuming all ports have the same } Z_0\text{)}$$

where $\boldsymbol{1}$ is the identity matrix (N by N)

$$\boldsymbol{Z} = \boldsymbol{Y}^{-1}$$

$$\boldsymbol{Y} = \boldsymbol{Z}^{-1}$$

Conversion between $\boldsymbol{S}$ and $\boldsymbol{Y}$ and $\boldsymbol{Z}$ (cont.)

$$\boldsymbol{S} = (\boldsymbol{Z}\boldsymbol{Y}_0 + \boldsymbol{1})^{-1}(\boldsymbol{Z}\boldsymbol{Y}_0 - \boldsymbol{1})$$

$$\boldsymbol{Z} = \boldsymbol{Y}_0^{-1}(\boldsymbol{1}+\boldsymbol{S})(\boldsymbol{1}-\boldsymbol{S})^{-1}$$

where $\boldsymbol{1}$ is the identity matrix (N by N) and

$$\boldsymbol{Y}_0 = diag(1/Z_{01} \quad 1/Z_{02} \quad \ldots \quad 1/Z_{0N})$$

$$\boldsymbol{Z} = \boldsymbol{Y}^{-1}$$

$$\boldsymbol{Y} = \boldsymbol{Z}^{-1}$$

Some Properties of the *S* Matrix

- If the network is reciprocal (passive and not containing any nonreciprocal media such as ferrites or plasmas), then ***S*** is symmetric,

$$S_{ij} = S_{ji}$$

- If the network is lossless, matrix ***S*** is unitary (orthonormal complex)

$$\sum_{k=1}^{N} S_{ki} S_{ki}^{*} = 1 \qquad \sum_{k=1}^{N} S_{ki} S_{kj}^{*} = 0$$

- In most practical cases, high-speed interconnects can be considered as reciprocal networks, and in some cases, lossless networks