Statistical Analysis and Yield Calculations

(Part 3)

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Outline

- Applications of sensitivities
- Sensitivity definitions
- Worst-Case analysis
- Using sensitivities to first-order statistical analysis

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Sensitivities

Sensitivity information is useful to:

- understand how the variation of circuit parameters affects a response
- asses the quality of different circuits having the same nominal response
- calculate gradients in optimization algorithms
- perform efficient Monte-Carlo analysis and yield prediction (if tolerances are small)

Sensitivity Definitions

- Circuit parameters: $\boldsymbol{y} \in \Re^t$
- Circuit responses: $\boldsymbol{R}(\boldsymbol{y}) \in \Re^r$
- Sensitivity of k-th response R_k with respect to the i-th parameter y_i can be defined in four ways:

Sensitivity

$$S_{y_i}^{R_k} = \frac{\partial R_k}{\partial y_i}$$

Normalized sensitivity

$$S_{y_i}^{R_k} = \frac{\partial \ln R_k}{\partial \ln y_i} = \frac{y_i}{R_k}\frac{\partial R_k}{\partial y_i}$$

Semi-normalized (w.r.t. y_i)

$$S_{y_i}^{R_k} = \frac{\partial R_k}{\partial \ln y_i} = y_i \frac{\partial R_k}{\partial y_i}$$

Semi-normalized (w.r.t. R_k)

$$S_{y_i}^{R_k} = \frac{\partial \ln R_k}{\partial y_i} = \frac{1}{R_k}\frac{\partial R_k}{\partial y_i}$$

Sensitivity Definitions (cont.)

- Since $R_k(\boldsymbol{y})$: $\Re^t \rightarrow \Re$, then

$$dR_k = \sum_{i=1}^{t} \frac{\partial R_k}{\partial y_i} dy_i \qquad \frac{dR_k}{R_k} = \sum_{i=1}^{t} S_{y_i}^{R_k} \frac{dy_i}{y_i}$$

with

$$S_{y_i}^{R_k} = \frac{\partial \ln R_k}{\partial \ln y_i} = \frac{y_i}{R_k} \frac{\partial R_k}{\partial y_i}$$

- The (relative) change in the k-th response R_k when all the circuit parameters simultaneously change by an small amount Δy_i can be estimated using:

$$\Delta R_k \approx \sum_{i=1}^{t} \frac{\partial R_k}{\partial y_i} \Delta y_i \qquad \frac{\Delta R_k}{R_k} = \sum_{i=1}^{t} S_{y_i}^{R_k} \frac{\Delta y_i}{y_i}$$

Sensitivity Definitions (cont.)

If equal tolerances and the same (normal) probability distribution functions are assumed for all the circuit parameters

$$MSS = \sqrt{\sum_{i=1}^{t} \left(S_{y_i}^{R_k}\right)^2}$$

MSS: Multi-parameter statistical sensitivity

with

$$S_{y_i}^{R_k} = \frac{\partial \ln R_k}{\partial \ln y_i} = \frac{y_i}{R_k} \frac{\partial R_k}{\partial y_i}$$

(J. Vlach and K. Singhal, 1983)

Worst Case Analysis

- The variation of the k-th response R_k with respect to all the circuit parameters is given by

$$dR_k = \sum_{i=1}^{t} \frac{\partial R_k}{\partial y_i} dy_i$$

- Assuming small changes in the parameters,

$$\Delta R_k \approx \sum_{i=1}^{t} \frac{\partial R_k}{\partial y_i} \Delta y_i = \boldsymbol{\Delta y}^{\mathrm{T}} \nabla R_k$$

- Assuming small tolerances τ_i for $i = 1 \ldots t$, the largest ΔR_k (worst case) is produced when

$$\Delta y_i = y_i \tau_i \operatorname{sign}\left(\frac{\partial R_k}{\partial y_i}\right)$$

Worst Case Analysis (cont.)

- The worst case responses can be estimated from the sensitivity information and tolerances by using

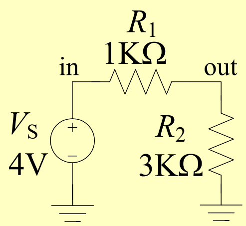
$$R_{k\,\min} = R_k - \Delta R_k$$
$$R_{k\,\max} = R_k + \Delta R_k$$

where

$$\Delta R_k = \boldsymbol{\Delta y}^{\mathrm{T}} \nabla R_k$$

$$\Delta y_i = y_i \tau_i \operatorname{sign}\left(\frac{\partial R_k}{\partial y_i}\right) \quad \text{for } i = 1 \ldots t$$

Worst Case Analysis – Example



$$\boldsymbol{y} = [V_{\rm S} \quad R_1 \quad R_2]^{\rm T}$$

$$\boldsymbol{y}^{\rm nom} = [4{\rm V} \quad 1{\rm K}\Omega \quad 3{\rm K}\Omega]^{\rm T}$$

$$\boldsymbol{\tau} = [0.03 \quad 0.05 \quad 0.1]^{\rm T}$$

$$\boldsymbol{R}(\boldsymbol{y}) = [V_{\rm in} \quad V_{\rm out} \quad I_{V_{\rm S}}]^{\rm T}$$

$$\boldsymbol{R}(\boldsymbol{y}^{\rm nom}) = [4{\rm V} \quad 3{\rm V} \quad 1{\rm mA}]^{\rm T}$$

Perform a worst case analysis for $V_{\rm out}$

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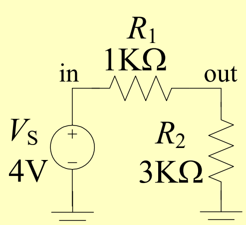
Worst Case Analysis – Example



$$v_o = \frac{V_S(R_2)}{(R_1 + R_2)}$$

Analytical sensitivities:

$$\frac{\partial v_o}{\partial R_1} = \frac{-V_S R_2}{(R_1+R_2)^2} = \frac{-4(3{\rm K})}{(4{\rm K})^2} = -0.75{\rm mV}/\Omega$$

$$\frac{\partial v_o}{\partial R_2} = \frac{-V_S(R_2)}{(R_1+R_2)^2} + \frac{V_S}{(R_1+R_2)} = \frac{V_S R_1}{(R_1+R_2)^2} = \frac{4(1{\rm K})}{(4{\rm K})^2} = +0.25{\rm mV}/\Omega$$

$$\frac{\partial v_o}{\partial V_S} = \frac{R_2}{R_1+R_2} = \frac{3{\rm K}}{4{\rm K}} = +0.75{\rm V/V}$$

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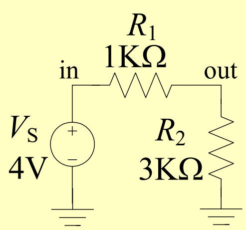
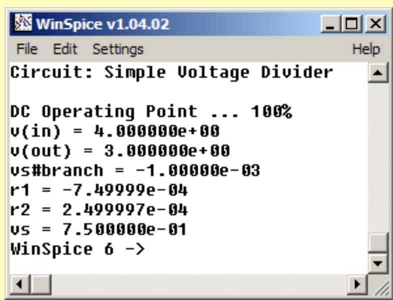
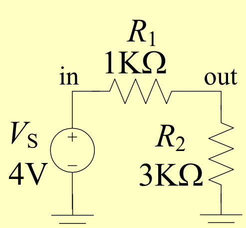
Worst Case Analysis – Example (cont.)



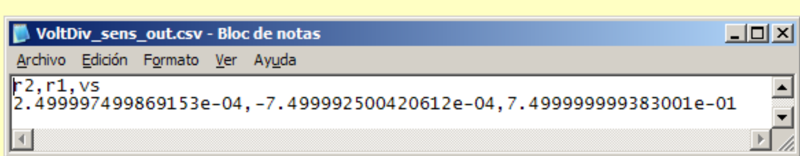

```
Simple Voltage Divider
*-----------------------
Vs    in   0    DC 4V
R1    in   out  1K
R2    out  0    3K
.control
op
print v(in) v(out) i(Vs)
sens v(out)
print all
.endc
.end
```



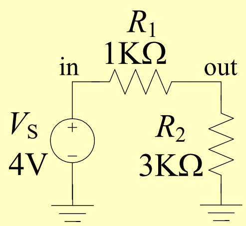
Worst Case Analysis – Example (cont.)




```
Simple Voltage Divider
*-----------------------
Vs    in   0    DC 4V
R1    in   out  1K
R2    out  0    3K
.control
op
write  VoltDiv_op_out.csv v(in) v(out) i(Vs)
sens v(out)
write VoltDiv_sens_out.csv
.endc
.end
```

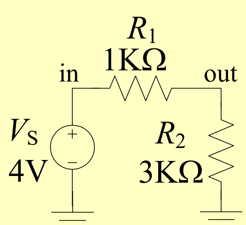


Worst Case Analysis – Example (cont.)



After generating a parameterized Matlab file (`VoltDiv_SPICE2`) to drive the SPICE simulation…

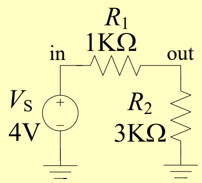
Worst Case Analysis – Example (cont.)



Estimating worst-case responses

```
% Nominal Design
Vs = 4; R1 = 1e3; R2 = 3e3; Ynom = [Vs R1 R2]';

% Calculating Responses and Sensitivites at Nominal Design
[psi,R] = VoltDiv_SPICE2(Ynom);

% Reading Responses and Sensitivities at Nominal Design
OPresp = R{1}; Vout = OPresp(:,3);
SENSresp = R{2}; SR2 = SENSresp(:,1); SR1 = SENSresp(:,2);
SVs = SENSresp(:,3);
Grad_Vout = [SVs SR1 SR2]'; % Gradient of the response.

% Define Tolerances
tauVs = 0.03; tauR1 = 0.05; tauR2 = 0.1;
tau = [tauVs tauR1 tauR2]'; % Vector of tolerances.

% Calculate Worst Case Parameter Changes
t = length(Ynom);
DeltaY = zeros(t,1);
for i = 1:t
    DeltaY(i) = Ynom(i)*(tau(i)*sign(Grad_Vout(i)));
end

% Calculate Worst Case Responses of Interest
Vout_WCp = Vout + DeltaY'*Grad_Vout;
Vout_WCn = Vout - DeltaY'*Grad_Vout;
```

Worst Case Analysis – Example (cont.)



$$\boldsymbol{y} = [V_S \quad R_1 \quad R_2]^T \qquad \boldsymbol{\tau} = [0.03 \quad 0.05 \quad 0.1]^T$$

Actual extreme responses:

$$v_{o\min} = \frac{V_{S\min}(R_{2\min})}{(R_{1\max} + R_{2\min})} \qquad v_{o\max} = \frac{V_{S\max}(R_{2\max})}{(R_{1\min} + R_{2\max})}$$

Worst-case estimations *vs* actual extreme values

$\boldsymbol{\tau}$ (%)	WC estimation (V)		actual values (V)		error (%)	
	v_{omin}	v_{omax}	v_{omin}	v_{omax}	(min)	(max)
$[12 \quad 20 \quad 40]^T$	2.19	3.81	2.112	3.7632	2.60	1.56
$[6 \quad 10 \quad 20]^T$	2.595	3.405	2.5783	3.392	0.557	0.433
$[3 \quad 5 \quad 10]^T$	2.7975	3.2025	2.7936	3.1991	0.130	0.115
$[1.5 \quad 2.5 \quad 5]^T$	2.8988	3.1012	2.8978	3.1004	0.031	0.029
$[0.75 \quad 1.25 \quad 2.5]^T$	2.9494	3.0506	2.9491	3.0504	0.0077	0.0075

Statistical Analysis by First-Order Model

- Circuit parameters: $\boldsymbol{y} \in \Re^t$
- Parameter tolerances: $\boldsymbol{\tau} \in \Re^t$
- Circuit responses: $\boldsymbol{R}(\boldsymbol{y}) \in \Re^r$
- Assuming small tolerances τ_i for $i = 1 \ldots t$, Monte-Carlo analysis and yield prediction can be approximated by using

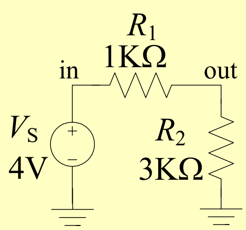
$$\boldsymbol{R}(\boldsymbol{y}_j) = \boldsymbol{R}(\boldsymbol{y}^{\text{nom}}) + \boldsymbol{\Delta R}_j \quad \text{for } j = 1 \ldots N$$

$$\Delta R_k = \boldsymbol{\Delta y}^T \nabla R_k \quad \text{for } k = 1 \ldots r$$

$$(\Delta y_i)_j = y_i \tau_i (2r_j - 1) \quad \text{for } i = 1 \ldots t$$

where r_j is a random number between 0 and +1 for each j

First-Order Statistical Analysis *vs* Monte Carlo



$$\boldsymbol{y} = [V_S \quad R_1 \quad R_2]^T$$

$$\boldsymbol{y}^{nom} = [4\text{V} \quad 1\text{K}\Omega \quad 3\text{K}\Omega]^T$$

$$\boldsymbol{\tau} = [0.03 \quad 0.05 \quad 0.1]^T$$

$$\boldsymbol{R}(\boldsymbol{y}) = [V_{in} \quad V_{out} \quad I_{V_S}]^T$$

$$\boldsymbol{R}(\boldsymbol{y}^{nom}) = [4\text{V} \quad 3\text{V} \quad 1\text{mA}]^T$$

Perform a first-order statistical analysis for V_{out} and compare versus Monte Carlo analysis

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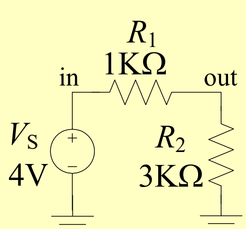
Monte-Carlo Analysis – Example



Assuming a uniform PDF for all y_i with $i = 1 \ldots t$

N circuit simulations

```
% Nominal Design
Vs = 4; R1 = 1e3; R2 = 3e3; Ynom = [Vs R1 R2];

% Calculate Responses of Interest at Nominal Design
[psi,R] = VoltDiv_SPICE1(Ynom);
OPresp = R{1};
Vout_nom = OPresp(:,3);

% Define Outcomes and Tolerances
N = 2000; % Number of outcomes.
tauVs = 0.03; % Tolerance for Vs.
tauR1 = 0.05; % Tolerance for R1.
tauR2 = 0.1;  % Tolerance for R2.
tau = [tauVs tauR1 tauR2]; % Vector of tolerances.

% Generate Random Outcomes with Uniform PDF
Y = zeros(N,length(Ynom)); % Matrix to store outcomes.
for j = 1:N
    Y(j,1) = Ynom(1)*(1+tau(1)*(2*rand-1));
    Y(j,2) = Ynom(2)*(1+tau(2)*(2*rand-1));
    Y(j,3) = Ynom(3)*(1+tau(3)*(2*rand-1));
end

% Calculate Responses of Interest at Random Outcomes
Vout_rand = zeros(1,N);
for j = 1:N
    [psi,R] = VoltDiv_SPICE1(Y(j,:));
    OPresp = R{1};
    Vout_rand(j) = OPresp(:,3);
end

% Plotting Results ...
```

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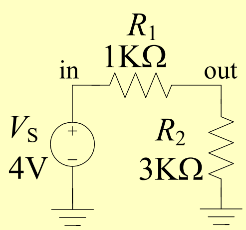
1^{st} Order Statistical Analysis – Example



Assuming a uniform PDF for all y_i with $i = 1 \ldots t$

```
% Nominal Design
Vs = 4; R1 = 1e3; R2 = 3e3; Ynom = [Vs R1 R2];

% Calculate Responses and Sensitivities at Nominal Design
[psi,R] = VoltDiv_SPICE2(Ynom);
OPresp = R{1}; Vout_nom = OPresp(:,3); SENSresp = R{2};
SR2 = SENSresp(:,1); SR1 = SENSresp(:,2); SVs = SENSresp(:,3);
Grad_Vout = [SVs SR1 SR2]'; % Gradient of the response.

% Define Outcomes and Tolerances
N = 2000; % Number of outcomes.
tauVs = 0.03; % Tolerance for Vs.
tauR1 = 0.05; % Tolerance for R1.
tauR2 = 0.1;  % Tolerance for R2.
tau = [tauVs tauR1 tauR2]; % Vector of tolerances.

% Generate Random Perturbations with Uniform PDF
DeltaY = zeros(N,length(Ynom)); % Matrix to store perturbations.
for j = 1:N
    DeltaY(j,1) = Ynom(1)*(tau(1)*(2*rand-1));
    DeltaY(j,2) = Ynom(2)*(tau(2)*(2*rand-1));
    DeltaY(j,3) = Ynom(3)*(tau(3)*(2*rand-1));
end

% Calculate Responses of Interest at Random Outcomes
Vout_rand = zeros(1,N);
for j = 1:N
    Vout_rand(j) = Vout_nom + DeltaY(j,:)*Grad_Vout;
end
```

No circuit simulations

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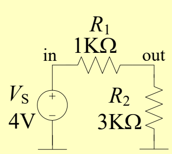
Statistical Analysis with $N = 2{,}000$

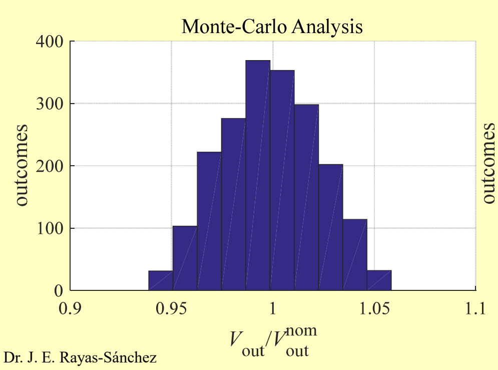


$$\boldsymbol{y} = [V_S \quad R_1 \quad R_2]^T$$

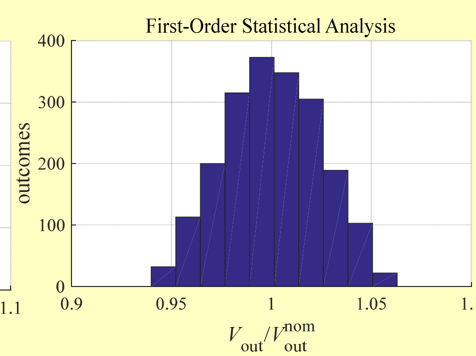
$$\boldsymbol{\tau} = [0.03 \quad 0.05 \quad 0.1]^T$$

Uniform PDF

Monte Carlo



First-Order Model



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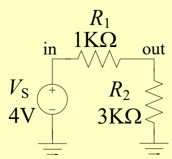
Statistical Analysis with $N = 2{,}000$ (cont.)

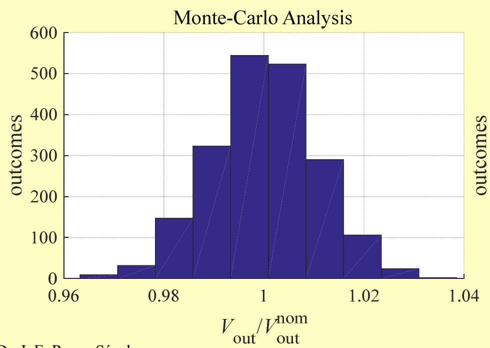


$\boldsymbol{y} = [V_S \quad R_1 \quad R_2]^T$

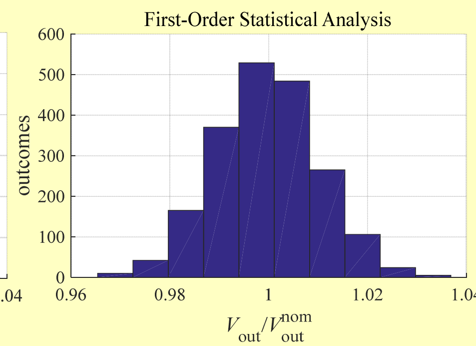
$\boldsymbol{\tau} = [0.03 \quad 0.05 \quad 0.1]^T$

Normal PDF

Monte Carlo



First-Order Model



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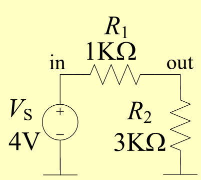
Yield Estimation



$\boldsymbol{y} = [V_S \quad R_1 \quad R_2]^T$

$\boldsymbol{\tau} = [0.03 \quad 0.05 \quad 0.1]^T$

Specs:

$2.9\text{V} \le V_{out} \le 3.1\text{V}$

Yield estimations with $N = 2{,}000$ outcomes and uniform PDF

method	yield (%)								
	run 1	run 2	run 3	run 4	run 5	run 6	run 7	μ_{yield}	σ_{yield}
Monte Carlo	83.25	82.7	82.9	80.55	81.55	81.9	83.7	82.4	1.0
1st order model	83.3	82.75	82.65	83.45	82.3	82.7	81.15	82.6	0.7

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