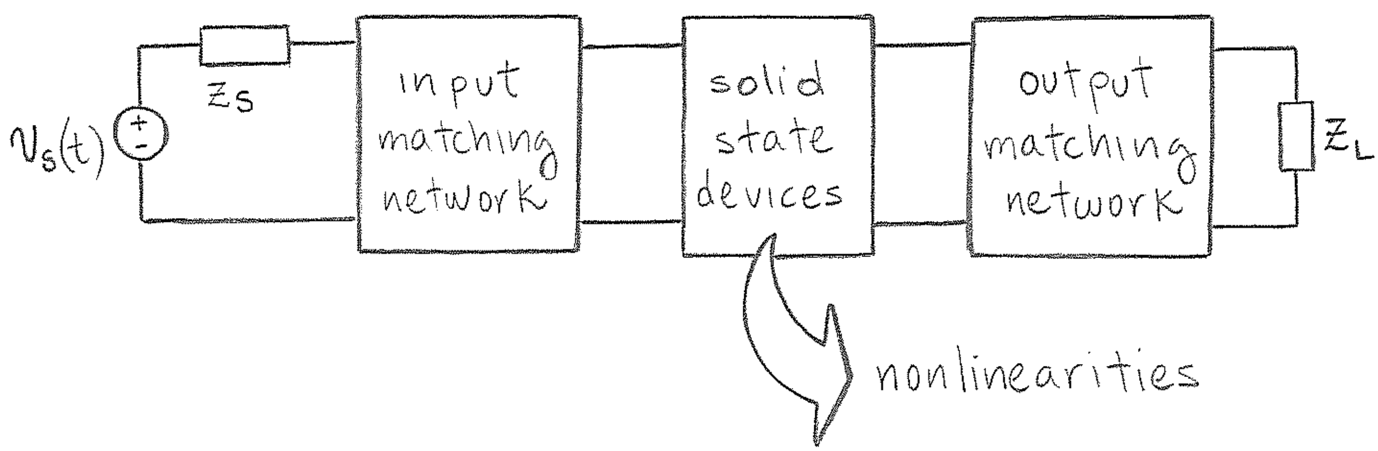
HARMONIC BALANCE

* Very useful for strongly nonlinear circuits with large signal excitation.

* Can be used for frequency domain or for steady state time domain analysis

General two-port microwave circuit:



$v_s(t)$ is typically a sinusoidal source with a DC bias component, but could be any waveform

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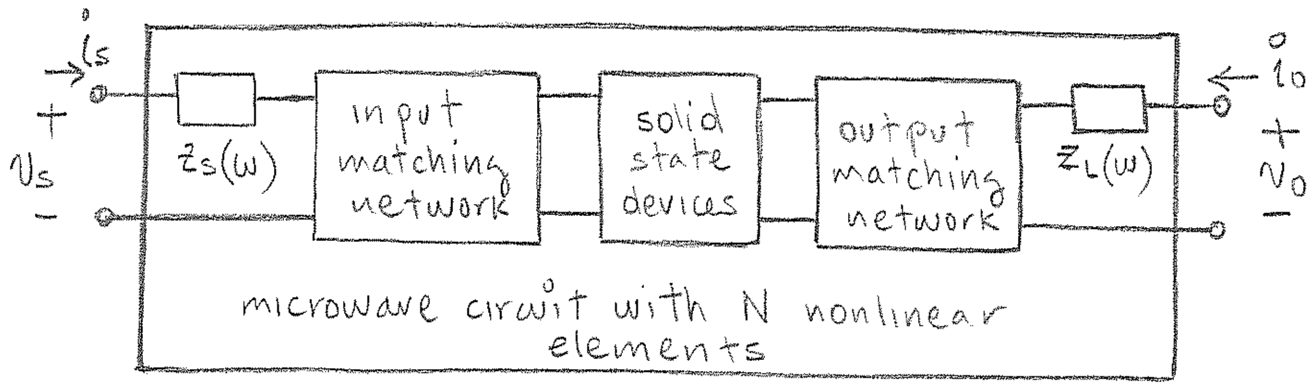
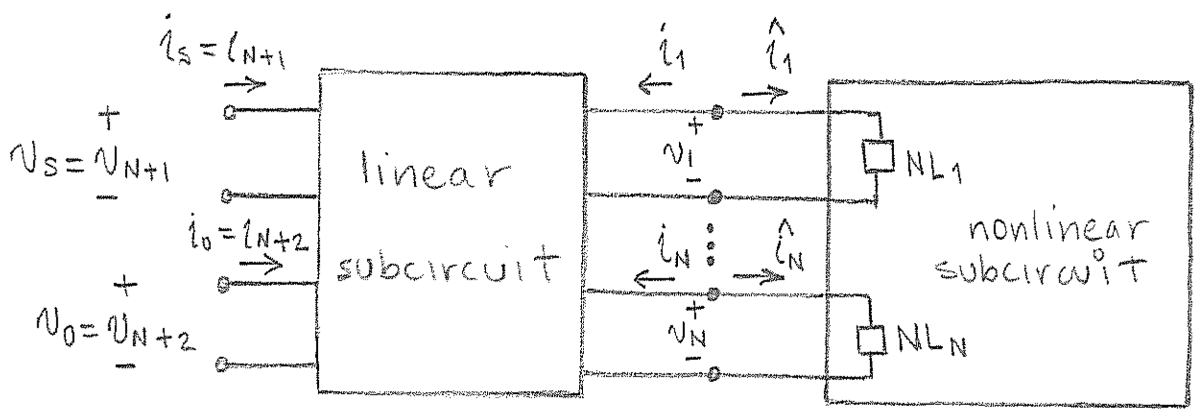
redrawing the microwave circuit:



$v_0(t)$ can be zero, or can be used to include any DC bias voltage

separating the linear and nonlinear elements:



NL_n is the n-th nonlinear element

HARMONIC BALANCE (HB) ⇒ find a set of waveforms (or harmonics) $v_1, \ldots v_N$ that give

$$i_n + \hat{i}_n = 0 \qquad n = 1, \ldots N$$

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Assuming that the DC component and the first H harmonics describe all voltages and currents adequately:

$$v_n = V_{n0} + \sum_{k=1}^{H} |V_{nk}| \cos(k\omega_p t + \theta_{nk})$$

$$i_n = I_{n0} + \sum_{k=1}^{H} |I_{nk}| \cos(k\omega_p t + \phi_{nk})$$

$$v_{N+1} = v_s(t) = V_{s0} + |V_{s1}| \cos(\omega_p t + \theta_s)$$

$$v_{N+2} = v_o(t) = V_{o0}$$

let $V_{nk} = |V_{nk}| \angle\theta_{nk}$, $I_{nk} = |I_{nk}| \angle\phi_{nk}$

$$\underset{\sim}{V_n} = \begin{bmatrix} V_{n0} \\ V_{n1} \\ \vdots \\ V_{nH} \end{bmatrix} \qquad \underset{\sim}{I_n} = \begin{bmatrix} I_{n0} \\ I_{n1} \\ \vdots \\ I_{nH} \end{bmatrix} \qquad \underset{\sim}{V} = \begin{bmatrix} \underset{\sim}{V_1} \\ \vdots \\ \underset{\sim}{V_N} \end{bmatrix} \qquad \underset{\sim}{I} = \begin{bmatrix} \underset{\sim}{I_1} \\ \vdots \\ \underset{\sim}{I_N} \end{bmatrix}$$

HB equation: $\underset{\sim}{I} + \underset{\sim}{\hat{I}} = \underset{\sim}{0}$

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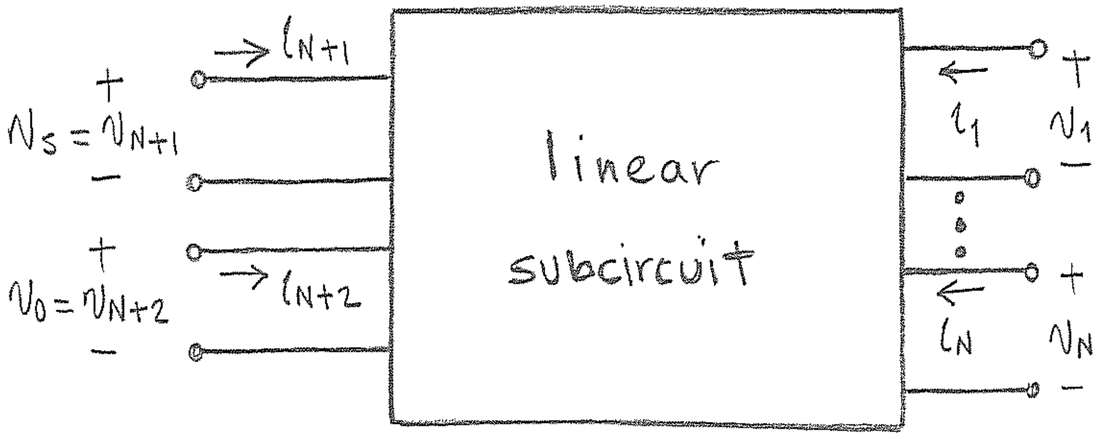
ANALYSIS OF THE LINEAR SUBCIRCUIT (Maas, 1997)



at the k-th harmonic:

$$\begin{bmatrix} I_{1k} \\ I_{2k} \\ \vdots \\ I_{N+2,k} \end{bmatrix} = \underset{\sim}{Y}_k (k\omega_p) \begin{bmatrix} V_{1k} \\ V_{2k} \\ \vdots \\ V_{N+2,k} \end{bmatrix}$$

$\underset{\sim}{Y}_k(k\omega_p)$ is the admittance matrix at the k-th harmonic

We can show that

$$\underset{\sim}{I} = \underset{\sim}{Y}\,\underset{\sim}{V} + \underset{\sim}{I}_p \quad \Rightarrow \text{currents in the linear subcircuit}$$

where $\underset{\sim}{Y}$ contains all the admittance matrices at the H harmonics

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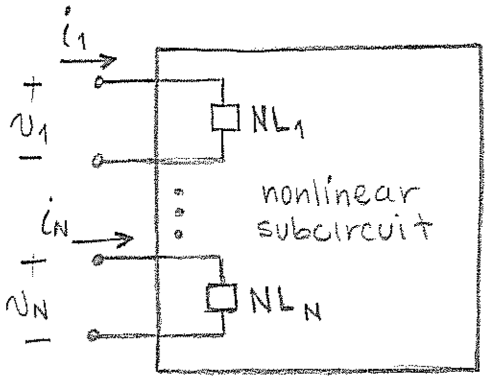
ANALYSIS OF THE NONLINEAR SUBCIRCUIT (Maas, 1997)



Since most of the nonlinearities are due to the solid state devices, we will consider nonlinear

* conductances
* controlled voltage current sources
* capacitors

* Nonlinear conductances and controlled sources

$$i_{G_n}(t) = g\left(v_1(t),\ v_2(t), \ldots, v_N(t)\right)$$

each voltage waveform can be obtained by Inverse-Fourier transforming the corresponding vector of harmonics

$$DFT^{-1}\{\underset{\sim}{V_n}\} \rightarrow v_n(t)$$

similarly

$$DFT\{i_{G_n}(t)\} \rightarrow \underset{\sim}{I}_{G_n} = \begin{bmatrix} I_{G_{n0}} \\ I_{G_{n1}} \\ \vdots \\ I_{G_{nH}} \end{bmatrix}$$

grouping

$$\underset{\sim}{I}_G = \begin{bmatrix} \underset{\sim}{I}_{G1} \\ \vdots \\ \underset{\sim}{I}_{GN} \end{bmatrix}$$

notice that

$$\underset{\sim}{I}_G = \underset{\sim}{I}_G(\underset{\sim}{V})$$

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* Nonlinear capacitors

We can show that

$$\underset{\sim}{I}_C = j\,\underset{\sim}{\Omega}\,\underset{\sim}{Q}$$

$\underset{\sim}{\Omega}$ contains all the harmonic frequencies

$\underset{\sim}{Q}$ contains the harmonic components of the charge at each nonlinear capacitor

HB eq. $\Rightarrow \quad \underset{\sim}{I} + \underset{\sim}{\hat{I}} = \underset{\sim}{0}$

$$\underset{\sim}{I} = \underset{\sim}{Y}\,\underset{\sim}{V} + \underset{\sim}{I}_p$$

$$\underset{\sim}{\hat{I}} = \underset{\sim}{I}_G(\underset{\sim}{V}) + \underset{\sim}{I}_C(\underset{\sim}{V})$$

$$\underset{\sim}{I}_C = j\,\underset{\sim}{\Omega}\,\underset{\sim}{Q}(\underset{\sim}{V})$$

HB equation: $\underset{\sim}{F}(\underset{\sim}{V}) = \underset{\sim}{Y}\,\underset{\sim}{V} + \underset{\sim}{I}_p + \underset{\sim}{I}_G(\underset{\sim}{V}) + j\,\underset{\sim}{\Omega}\,\underset{\sim}{Q}(\underset{\sim}{V}) = \underset{\sim}{0}$

$\underset{\sim}{F}(\underset{\sim}{V}) \Rightarrow$ difference between the currents calculated from the linear and nonlinear subcircuit, at each port, at each harmonic.

solving $\underset{\sim}{F}(\underset{\sim}{V}) = \underset{\sim}{0}$ gives the frequency domain behavior of the nonlinear network

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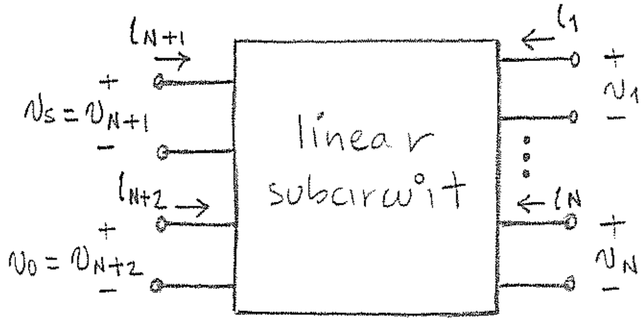
ANALYSIS OF THE LINEAR SUBCIRCUIT (more details)



at the k-th harmonic:

$$\begin{bmatrix} I_{1k} \\ I_{2k} \\ \vdots \\ I_{N+2,k} \end{bmatrix} = \underset{\sim}{Y}_k(k\omega_p) \begin{bmatrix} V_{1k} \\ V_{2k} \\ \vdots \\ V_{N+2,k} \end{bmatrix}$$

$\underset{\sim}{Y}_k(k\omega_p)$: admitance matrix at the k-th harmonic

$$y_{nm}(k\omega_p) = \left. \frac{I_{nk}}{V_{mk}} \right|_{\substack{V_{jk}=0 \\ j \neq m}} \qquad \begin{array}{l} n = 1, \dots N+2 \\ m = 1, \dots N+2 \end{array}$$

Since

$$I_{nk} = y_{n1}(k\omega_p)V_{1k} + y_{n2}(k\omega_p)V_{2k} + \dots + y_{n,N+2}(k\omega_p)V_{N+2,k}$$

$$\underset{\sim}{I_n} = \begin{bmatrix} I_{n0} \\ I_{n1} \\ \vdots \\ I_{nH} \end{bmatrix} = \begin{bmatrix} y_{n1}(0)V_{10} + y_{n2}(0)V_{20} + \dots + y_{n,N+2}(0)V_{N+2,0} \\ y_{n1}(\omega_p)V_{11} + y_{n2}(\omega_p)V_{21} + \dots + y_{n,N+2}(\omega_p)V_{N+2,1} \\ \vdots \\ y_{n1}(H\omega_p)V_{1H} + y_{n2}(H\omega_p)V_{2H} + \dots + y_{n,N+2}(H\omega_p)V_{N+2,H} \end{bmatrix}$$

Let $\underset{\sim}{I_n} = \underset{\sim}{Y_{n,1}}\,\underset{\sim}{V_1} + \underset{\sim}{Y_{n,2}}\,\underset{\sim}{V_2} + \dots + \underset{\sim}{Y_{n,N+2}}\,\underset{\sim}{V_{N+2}}$

where $\underset{\sim}{Y_{n,m}} = \mathrm{diag}\left(y_{nm}(k\omega_p)\right) \qquad k = 0, 1, \dots H$

$\underset{\sim}{Y_{n,m}}$ contains elements y_{nm} at all freq. components

Then

$$\begin{bmatrix} \underset{\sim}{I}_1 \\ \underset{\sim}{I}_2 \\ \vdots \\ \underset{\sim}{I}_{N+2} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{Y}_{1,1} & \underset{\sim}{Y}_{1,2} & \cdots & \underset{\sim}{Y}_{1,N+2} \\ \underset{\sim}{Y}_{2,1} & \underset{\sim}{Y}_{2,2} & \cdots & \underset{\sim}{Y}_{2,N+2} \\ \vdots & & & \\ \underset{\sim}{Y}_{N+2,1} & \cdots & & \underset{\sim}{Y}_{N+2,N+2} \end{bmatrix} \begin{bmatrix} \underset{\sim}{V}_1 \\ \underset{\sim}{V}_2 \\ \vdots \\ \underset{\sim}{V}_{N+2} \end{bmatrix}$$

The current in the firts N ports are

$$\begin{bmatrix} \underset{\sim}{I}_1 \\ \underset{\sim}{I}_2 \\ \vdots \\ \underset{\sim}{I}_N \end{bmatrix} = \begin{bmatrix} \underset{\sim}{Y}_{1,1} & \underset{\sim}{Y}_{1,2} & \cdots & \underset{\sim}{Y}_{1,N} \\ \underset{\sim}{Y}_{2,1} & \underset{\sim}{Y}_{2,2} & \cdots & \underset{\sim}{Y}_{2,N} \\ \vdots & & & \\ \underset{\sim}{Y}_{N,1} & \underset{\sim}{Y}_{N,2} & \cdots & \underset{\sim}{Y}_{N,N} \end{bmatrix} \begin{bmatrix} \underset{\sim}{V}_1 \\ \underset{\sim}{V}_2 \\ \vdots \\ \underset{\sim}{V}_N \end{bmatrix} + \begin{bmatrix} \underset{\sim}{Y}_{1,N+1} & \underset{\sim}{Y}_{1,N+2} \\ \vdots & \\ \underset{\sim}{Y}_{N,N+1} & \underset{\sim}{Y}_{N,N+2} \end{bmatrix} \begin{bmatrix} \underset{\sim}{V}_{N+1} \\ \underset{\sim}{V}_{N+2} \end{bmatrix}$$

$$\underset{\sim}{I} = \underset{\sim}{Y} \, \underset{\sim}{V} + \underset{\sim}{I}_p$$

currents in the linear
subcircuit

* Nonlinear capacitors (more details)

$$q_n(t) = f_n\left(v_1(t),\ v_2(t), \ldots\ v_N(t)\right)$$

where $\quad v_n(t) \leftarrow \mathrm{DFT}^{-1}\{ \underset{\sim}{V}_n \}$

taking $\quad \mathrm{DFT}\{ q_n(t) \} \rightarrow \underset{\sim}{Q}_n = \begin{bmatrix} Q_{n0} \\ Q_{n1} \\ \vdots \\ Q_{nH} \end{bmatrix}$

grouping in a single charge vector

$$\underset{\sim}{Q} = \left[\underset{\sim}{Q}_1^T \ \underset{\sim}{Q}_2^T \cdots \underset{\sim}{Q}_N^T \right]^T \qquad \left(\underset{\sim}{Q} = \underset{\sim}{Q}(\underset{\sim}{V}) \right)$$

The nonlinear capacitor current $i_c(t)$ at the n-th port is

$$i_{cn}(t) = \frac{d\, q_n(t)}{dt} \qquad \text{then}$$

$$\underset{\sim}{I}_{cn} = \begin{bmatrix} I_{cn0} \\ I_{cn1} \\ \vdots \\ I_{cnH} \end{bmatrix} = \begin{bmatrix} 0 \\ j\omega_p Q_{n1} \\ \vdots \\ jH\omega_p Q_{nH} \end{bmatrix} = j \begin{bmatrix} 0 & & & \\ & \omega_p & & \underset{\sim}{0} \\ & & 2\omega_p & \\ & \underset{\sim}{0} & & \ddots \\ & & & & H\omega_p \end{bmatrix} \underset{\sim}{Q}_n = j\underset{\sim}{\omega}\,\underset{\sim}{Q}_n$$

grouping in a single capacitor current vector $\underset{\sim}{I}_c$

$$\underset{\sim}{I}_c = \begin{bmatrix} \underset{\sim}{I}_{c1} \\ \vdots \\ \underset{\sim}{I}_{cN} \end{bmatrix}, \quad \text{if} \quad \underset{\sim}{\Omega} = \begin{bmatrix} \underset{\sim}{\omega} & & \\ & \underset{\sim}{\omega} & \\ & & \ddots \\ & & & \underset{\sim}{\omega} \end{bmatrix} \quad \text{then} \quad \underset{\sim}{I}_c = j\underset{\sim}{\Omega}\,\underset{\sim}{Q}$$