Formulation of Circuit Equations

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Introduction

- A physical circuit is modeled by a number of linear and nonlinear circuit elements connected in a given configuration
- Its behavior is governed by two kinds of equations:
  - Equations for describing the operation of each circuit-element (I-V relationship for each branch)
  - Equations for describing how the circuit-elements are connected (topology)

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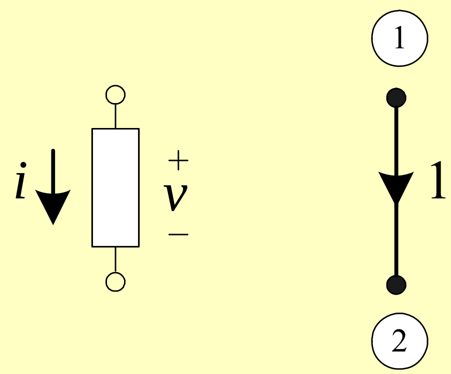
Circuit Theory – EM Theory

- Voltages and currents cannot be uniquely defined for transmission lines that do not support transverse electromagnetic (TEM) waves
- Classic circuit theory is valid as a quasi-static approximation of general electromagnetic theory
- Kirchhoff´s laws are still applicable at high frequencies provided that
  - Distributed-circuit and radiation effects are modeled within circuit elements (they are treated as "black boxes")
  - Voltages and currents are defined at the terminals of every circuit element

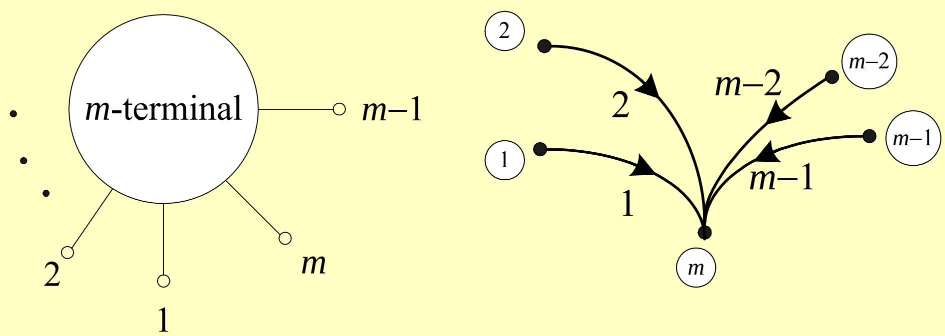
Oriented Graphs

- It is a graphical representation of the nodes and branches of a circuit
- Branches are oriented by means of an arrow
- A graph is connected if there is at least one path made of branches between every pair of nodes
- An oriented graph retains the topological information about a circuit, suppressing the information about the circuit elements

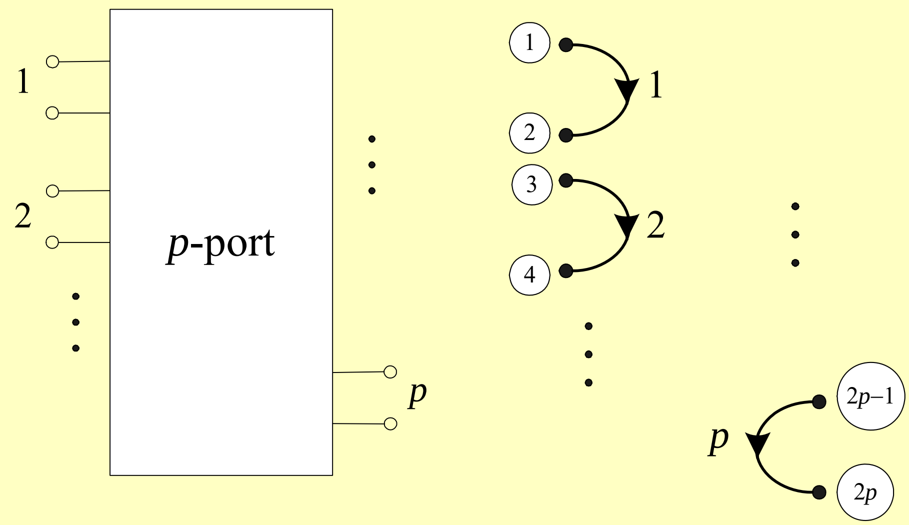
Two-Terminal Elements



Multi-Terminal Elements



Multi-Port Elements



Notation

- Scalars: a, b, c
- Vectors: $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{V}$
- Matrices: $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$
- Handwriting…
- Identity matrix: $\boldsymbol{I}$ or **1**
- Examples:

$$a \in \Re \quad \boldsymbol{b} \in \Re^n \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \qquad \boldsymbol{C} \in \Re^{n \times m} \quad \boldsymbol{C} = \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nm} \end{bmatrix}$$

Incidence Matrix $\boldsymbol{A}$

- It stores the topology information of the circuit
- If the circuit has n+1 nodes and b branches,

$$\boldsymbol{A} \in \Re^{n \times b} \qquad \boldsymbol{A} = [a_{ij}]$$

$$a_{ij} = \begin{cases} 1 & \text{if branch } j \text{ leaves node } i \\ -1 & \text{if branch } j \text{ enters node } i \\ 0 & \text{if branch } j \text{ does not touch node } i \end{cases}$$

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Kirchhoff's Current Law (KCL)

If $\boldsymbol{I}$ is the vector of branch currents,

$$\boldsymbol{I} \in \Re^{b \times 1} \qquad \boldsymbol{I} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_b \end{bmatrix}$$

$$\boldsymbol{AI} = \boldsymbol{0}$$

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Kirchhoff's Voltage Law (KVL)

If $\boldsymbol{V}_n$ is the vector of node voltages, and $\boldsymbol{V}_b$ is the vector of branch voltages,

$$\boldsymbol{V}_n \in \Re^{n\times 1} \qquad \boldsymbol{V}_b \in \Re^{b\times 1} \qquad \boldsymbol{V}_n = \begin{bmatrix} V_{n1} \\ V_{n2} \\ \vdots \\ V_{nn} \end{bmatrix} \qquad \boldsymbol{V}_b = \begin{bmatrix} V_{b1} \\ V_{b2} \\ \vdots \\ V_{bb} \end{bmatrix}$$

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{V}_n - \boldsymbol{V}_b = \boldsymbol{0}$$

$$\boldsymbol{V}_b = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{V}_n$$

Tableau Formulation

- $\boldsymbol{I}$, $\boldsymbol{V}_b$ and $\boldsymbol{V}_n$ are taken as the unkowns
- For a circuit that corresponds to a connected graph with b branches and n+1 nodes, the number of unknowns is 2b+n
- Equations

$$\boldsymbol{AI} = \boldsymbol{0} \qquad \boldsymbol{A}^{\mathrm{T}}\boldsymbol{V}_n - \boldsymbol{V}_b = \boldsymbol{0}$$

  provide $n + b$ linearly independent equations
- The rest of the equations (b) must be obtained from the circuit element definitions

Tableau Formulation (cont.)

General representation of a linear circuit element

$$\begin{bmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{K}_2 \end{bmatrix} \boldsymbol{V}_b + \begin{bmatrix} \boldsymbol{K}_1 \\ \boldsymbol{Z}_2 \end{bmatrix} \boldsymbol{I} = \begin{bmatrix} \boldsymbol{W}_1 \\ \boldsymbol{W}_2 \end{bmatrix}$$

where

$\boldsymbol{Y}_1$ and $\boldsymbol{Z}_2$ represent admittances and impedances, respectively;

$\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ contain dimensionless constants;

and $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ include the independent current and voltage sources, as well as the influence of initial conditions on capacitors and inductors

Tableau Formulation (cont.)

Using a more compact notation,

$$\boldsymbol{Y}\boldsymbol{V}_b + \boldsymbol{Z}\boldsymbol{I} = \boldsymbol{W}$$

where

$\boldsymbol{Y} = [\boldsymbol{Y}_1 \;\; \boldsymbol{K}_2]^{\mathrm{T}}$, $\boldsymbol{Z} = [\boldsymbol{K}_1 \;\; \boldsymbol{Z}_2]^{\mathrm{T}}$ and $\boldsymbol{W} = [\boldsymbol{W}_1 \;\; \boldsymbol{W}_2]^{\mathrm{T}}$

$\boldsymbol{Y} \in \Re^{b\times b}$ $\quad \boldsymbol{Z} \in \Re^{b\times b}$ $\quad \boldsymbol{W} \in \Re^{b}$

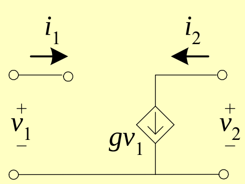
Selecting *Y*, *Z* and *W*: Simple Elements

Element	Equation	$\boldsymbol{Y}$	$\boldsymbol{Z}$	$\boldsymbol{W}$
Resistor	$V - RI = 0$	1	$-R$	0
Conductor	$GV - I = 0$	G	-1	0
Capacitor	$sCV - I = CV_0$	sC	-1	CV_0
Inductor	$V - sLI = -LI_0$	1	$-sL$	$-LI_0$
Voltage source	$V = E$	1	0	E
Current source	$I = J$	0	1	J

$$(\boldsymbol{YV}_b + \boldsymbol{ZI} = \boldsymbol{W})$$

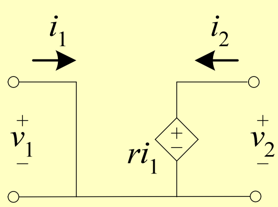
Selecting *Y*, *Z* and *W*: Controlled Sources

VCCS (Voltage-Controlled Current Source)



$$\begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

CCVS (Current-Controlled Voltage Source)



$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -r & 0 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Etc.

$$(\boldsymbol{YV}_b + \boldsymbol{ZI} = \boldsymbol{W})$$

Selecting ***Y***, ***Z*** and ***W***: Parameters Z and Y

Any multi-port network modeled by classical impedance or admittance parameters can also be represented by

$$\boldsymbol{Y}\boldsymbol{V}_b + \boldsymbol{Z}\boldsymbol{I} = \boldsymbol{W}$$

If the network is modeled by impedance parameters ***Z***:

$$\boldsymbol{V}_b = \boldsymbol{Z}\boldsymbol{I}, \boldsymbol{Y} = -\mathbf{1}, \boldsymbol{W} = \mathbf{0}$$

If the network is modeled by admittance parameters ***Y***:

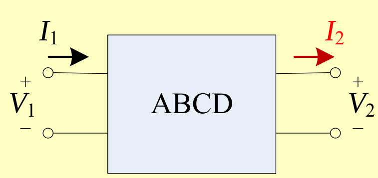
$$\boldsymbol{I} = \boldsymbol{Y}\boldsymbol{V}_b, \boldsymbol{Z} = -\mathbf{1}, \boldsymbol{W} = \mathbf{0}$$

Selecting ***Y***, ***Z*** and ***W***: Parameters ABCD

Any two-port network modeled by classical ABCD parameters can also be represented by

$$\boldsymbol{Y}\boldsymbol{V}_b + \boldsymbol{Z}\boldsymbol{I} = \boldsymbol{W}$$

If the network is modeled by ABCD parameters:



$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

then

$$\begin{bmatrix} -1 & A \\ 0 & C \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 & -B \\ -1 & -D \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Tableau Formulation (cont.)

Combining KVL, KCL and circuit elements,

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{V}_n - \boldsymbol{V}_b = \boldsymbol{0}$$

$$\boldsymbol{Y}\boldsymbol{V}_b + \boldsymbol{Z}\boldsymbol{I} = \boldsymbol{W}$$

$$\boldsymbol{A}\boldsymbol{I} = \boldsymbol{0}$$

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & -\boldsymbol{A}^{\mathrm{T}} \\ \boldsymbol{Y} & \boldsymbol{Z} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_b \\ \boldsymbol{I} \\ \boldsymbol{V}_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{W} \\ \mathbf{0} \end{bmatrix}$$

Tableau equation:

$$\boldsymbol{T}\boldsymbol{X} = \boldsymbol{W}'$$

$$\boldsymbol{T} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & -\boldsymbol{A}^{\mathrm{T}} \\ \boldsymbol{Y} & \boldsymbol{Z} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A} & \mathbf{0} \end{bmatrix} \qquad \boldsymbol{W}' = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{W} \\ \mathbf{0} \end{bmatrix}$$

$$\boldsymbol{T} \in \Re^{(2b+n)\times(2b+n)}$$

$$\boldsymbol{X} \in \Re^{(2b+n)\times 1}$$

$$\boldsymbol{W}' \in \Re^{(2b+n)\times 1}$$

Matrix Density D

The density D of a matrix is defined as

$$D = \frac{\text{number of nonzero entries in the matrix}}{\text{total number of all entries in the matrix}}$$

Typically, $D < 10\%$ for $\boldsymbol{T}$ (very sparse matrix)