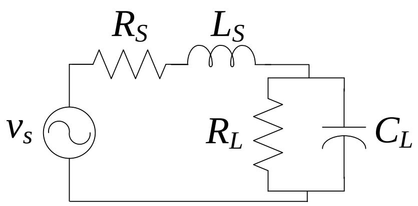
CONSTRAINED OPTIMIZATION

Dr. J. E. Rayas-Sánchez

GRAPHICAL EXAMPLES OF UNCONSTRAINED AND CONSTRAINED OPTIMIZATION PROBLEMS

1. Maximizing Power Transfer in a Simple AC Circuit – Unconstrained Problem

Consider the following AC circuit. Assuming that R_S = 50 Ω, L_S = 5 nH, v_s = (1V)sin2πft, with f = 1GHz, we want to find the optimal values of R_L and C_L that maximize the power delivered to the load.



From basic circuit theory we know that this problem has a closed form solution. The power delivered to the load is maximum when $Z_S = Z_L^*$ (load impedance equal to the complex conjugate of the source impedance). Using this fact, the optimal values of R_L and C_L at 1GHz are

R_L optimum = 69.7392 Ω
C_L optimum = 1.4339 pF

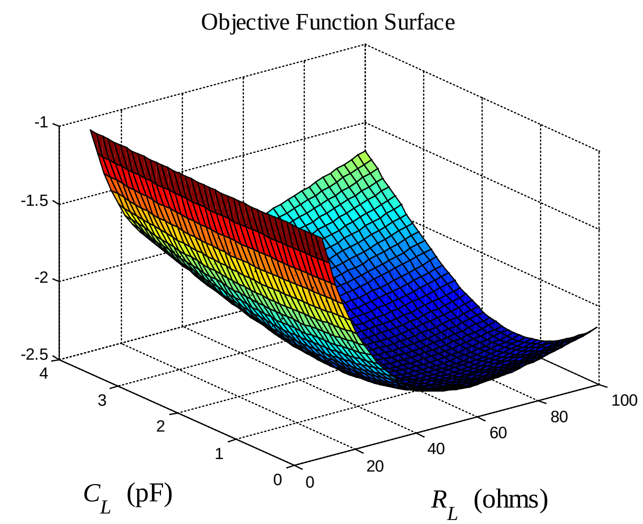
For which the corresponding impedances are

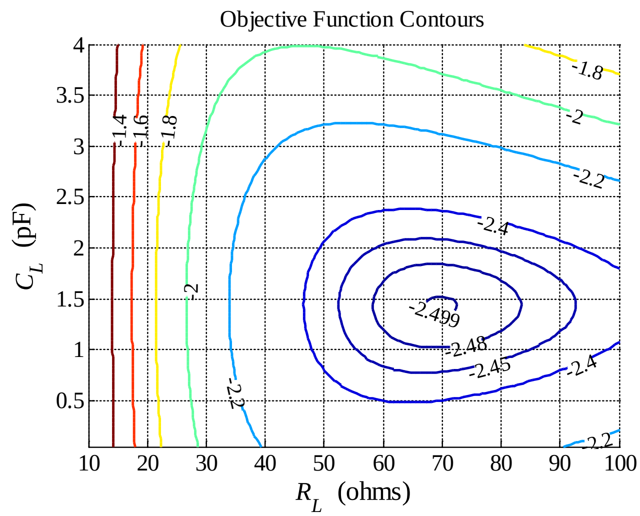
Z_S = 50 + j31.4159 Ω
Z_L = 50 – j31.4159 Ω

and the average real power delivered to the load is

$P_{L\max}$ = 2.5 mW

The corresponding optimization problem is $\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} u(\boldsymbol{x})$. The optimization variables are $\boldsymbol{x} = [R_L \quad C_L]^T$. Since we want to maximize the power at the load, $u(\boldsymbol{x}) = -P_L$, where $P_L = \frac{1}{2}\text{Re}\{V_L I_L^*\}$ and $V_L(\boldsymbol{x}) = I_L Z_L$, $I_L(\boldsymbol{x}) = \dfrac{V_S}{Z_S + Z_L}$, $Z_S = R_S + j2\pi f L_S$, $Z_L = \dfrac{R_L}{1 + j2\pi f R_L C_L}$.





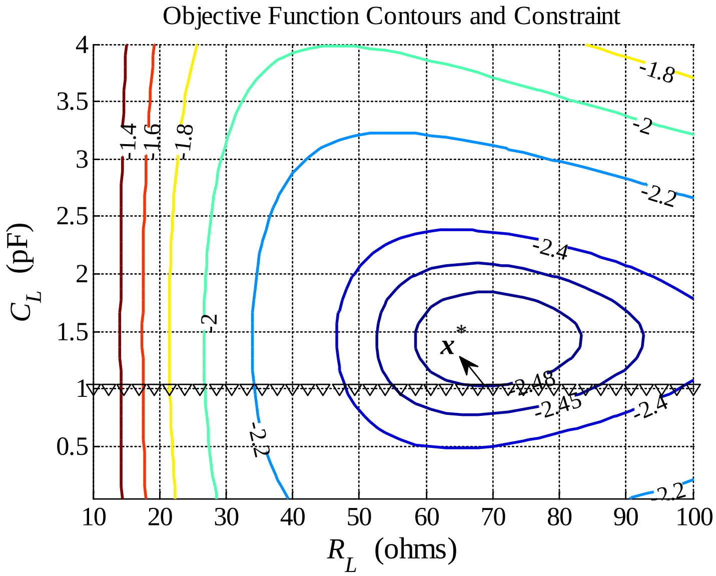
It is seen that the graphical solution of the optimization problem agrees with the theoretical solution.

2. Maximizing Power Transfer in a Simple AC Circuit – Adding Box Constraints

Let us assume that we want to solve the same problem but now considering a maximum load capacitance $C_{L\max}$ = 1 pF. The new optimization problem is

$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} u(\boldsymbol{x}) \quad \text{subject to} \quad x_2 \le C_{L\max}$$

with $u(\boldsymbol{x}) = -P_L$, $\boldsymbol{x} = [R_L \quad C_L]^T$, $P_L = \frac{1}{2}\mathrm{Re}\{V_L I_L^*\}$, $V_L(\boldsymbol{x}) = I_L Z_L$, $I_L(\boldsymbol{x}) = \dfrac{V_S}{Z_S + Z_L}$, $Z_S = R_S + j2\pi f L_S$, and $Z_L(\boldsymbol{x}) = \dfrac{R_L}{1 + j2\pi f R_L C_L}$.

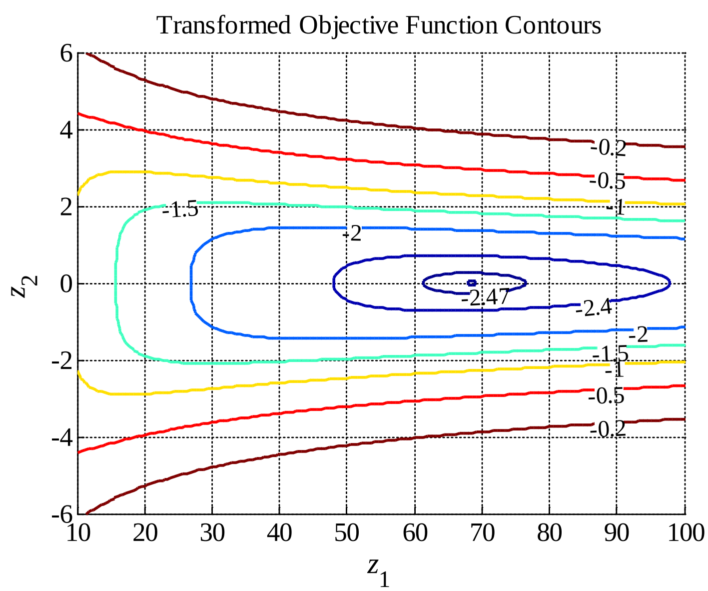


Zooming in the graphical solution it is seen that $\boldsymbol{x}^* \approx [68.5\Omega \quad 1\text{pF}]^T$.

We can remove the box constraints through variable transformations, as follows,

$$\boldsymbol{z}^* = \arg\min_{\boldsymbol{z}} u(\boldsymbol{z}) \qquad \text{where } u(\boldsymbol{z}) = u(\boldsymbol{x}) \text{ with } x_1 = z_1 \text{ and } x_2 = C_{L\max} - z_2^2.$$

The contours of the new unconstrained objective are:



Zooming in the graphical solution it is seen that $\boldsymbol{z}^* \approx [68.5\Omega \quad 0]^T$, which corresponds to $\boldsymbol{x}^* \approx [68.5\Omega \quad 1\text{pF}]^T$.

The optimal response at the constrained solution is $P_{L\max}$ = 2.4778 mW.

3. Maximizing Power Transfer in a Simple AC Circuit – Adding Inequality Constraints

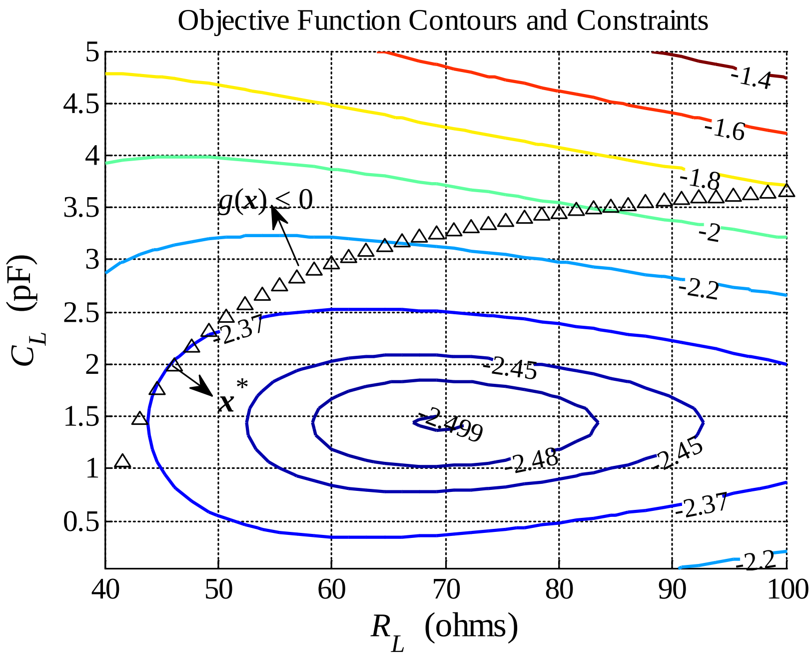
Let us assume that we want to solve the same problem but now restricted to a magnitude of the load admittance larger than 25mΩ^{-1}, that is, $|Y_L| \geq Y_{L\min} = 25\text{m}\Omega^{-1}$. The new optimization problem is

$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} u(\boldsymbol{x}) \quad \text{subject to} \quad \boldsymbol{g}(\boldsymbol{x}) \leq 0$$

with $u(\boldsymbol{x}) = -P_L$, $\boldsymbol{x} = [R_L \quad C_L]^T$, $P_L = \frac{1}{2}\text{Re}\{V_L I_L^*\}$, $V_L(\boldsymbol{x}) = I_L Z_L$,

$I_L(\boldsymbol{x}) = \dfrac{V_S}{Z_S + Z_L}$, $Z_S = R_S + j2\pi f L_S$, $Z_L(\boldsymbol{x}) = \dfrac{R_L}{1 + j2\pi f R_L C_L}$, and

$\boldsymbol{g}(\boldsymbol{x}) = Y_{L\min} - |1/Z_L(\boldsymbol{x})|$, $Y_{L\min} = 25\text{m}\Omega^{-1}$.



Zooming in the graphical solution it is seen that $\boldsymbol{x}^* \approx [46.1\Omega \quad 1.97\text{pF}]^T$.

The above problem can be solved as an unconstrained optimization problem using penalty functions. The new objective function is

$$U(\boldsymbol{x}) = u(\boldsymbol{x}) + r^g (G(\boldsymbol{x}))^2$$

with $u(\boldsymbol{x}) = -P_L$, $\boldsymbol{x} = [R_L \quad C_L]^T$, $P_L = \frac{1}{2}\text{Re}\{V_L I_L^*\}$, $V_L(\boldsymbol{x}) = I_L Z_L$,

$I_L(\boldsymbol{x}) = \dfrac{V_S}{Z_S + Z_L}$, $Z_S = R_S + j2\pi f L_S$, $Z_L(\boldsymbol{x}) = \dfrac{R_L}{1 + j2\pi f R_L C_L}$, and

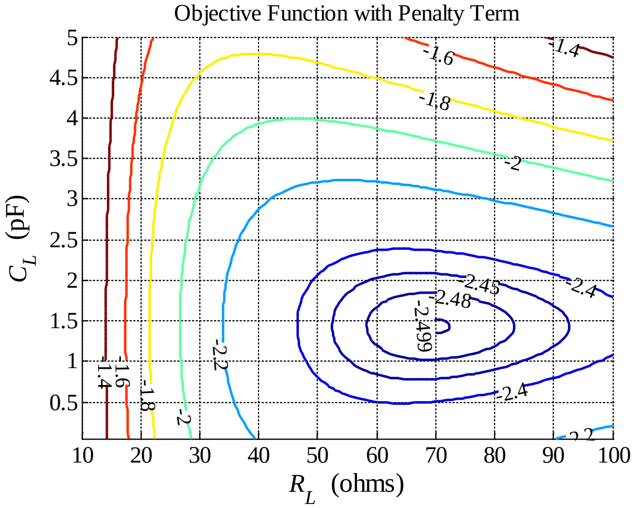
$G = \max\{0, g(\boldsymbol{x})\}$,

$g(\boldsymbol{x}) = Y_{L\min} - |1/Z_L(\boldsymbol{x})|$,

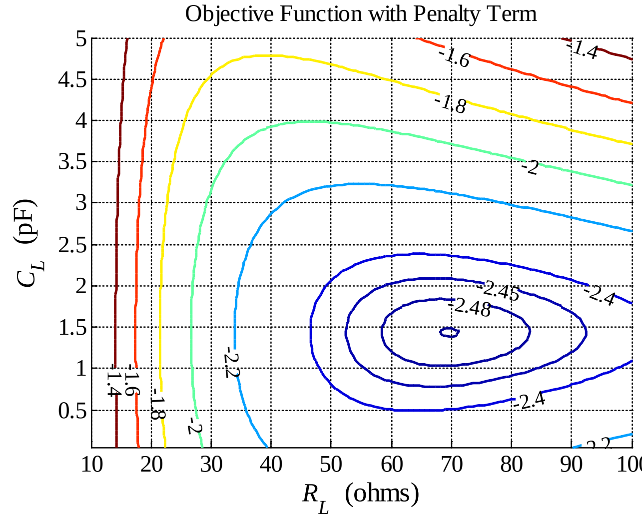
$Y_{L\min} = 25\text{m}\Omega^{-1}$.

The optimal solution found, $\boldsymbol{x}^*$, depends on the value of the penalty term r^g, as illustrated in the following contours of $U(\boldsymbol{x})$:

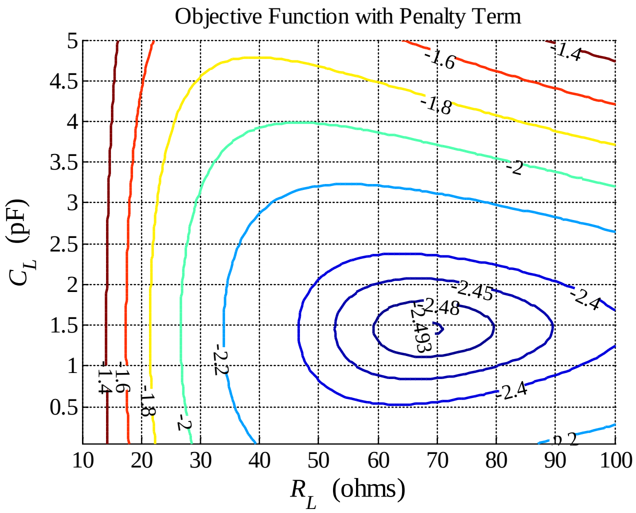
$r^g = 1$



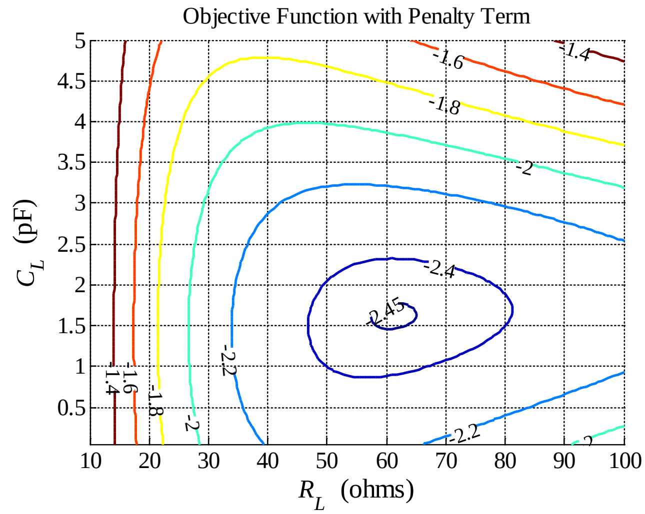
$r^g = 10$



$r^g = 100$



$r^g = 1000$



The values used above for the penalty coefficient r^g are too small. This makes that the overall objective function $U(\boldsymbol{x})$ does not "see" the effects of the inequality constraint $g(\boldsymbol{x})$.

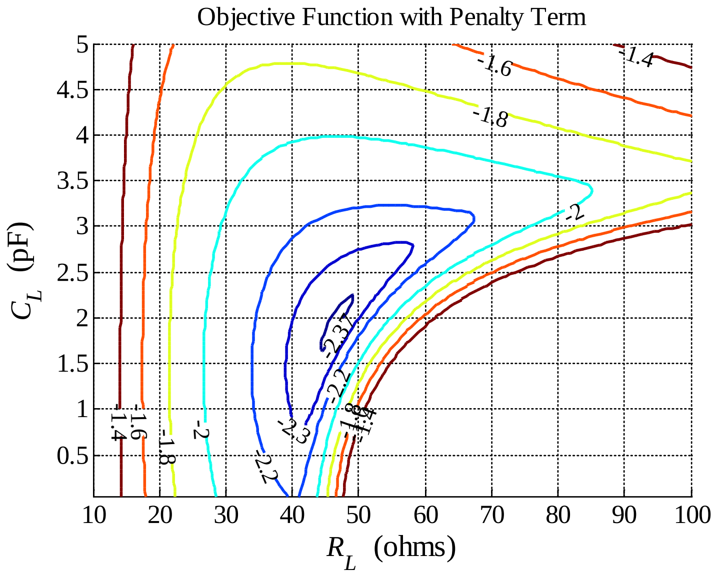
This problem illustrates how an arbitrary selection of the initial penalty coefficient r^g can yield an important amount of unnecessary unconstrained optimizations before we reach the solution.

The effect of selecting a better initial penalty coefficient r^g is now illustrated with two starting points.

a) Let us assume that the starting point is $\boldsymbol{x}_0 \approx [90\Omega \quad 4.5\text{pF}]^T$ (an interior point). Then $u(\boldsymbol{x}_0) = -1.5638$, $g(\boldsymbol{x}_0) = -0.0054$. A better way to choose the initial r^g is

$$r_0^g = \frac{|u(\boldsymbol{x}_0)|}{\|\boldsymbol{g}(\boldsymbol{x}_0)\|_2^2} = \frac{1.5638}{(0.0054)^2} = 53{,}628$$

Using r^g = 53,628

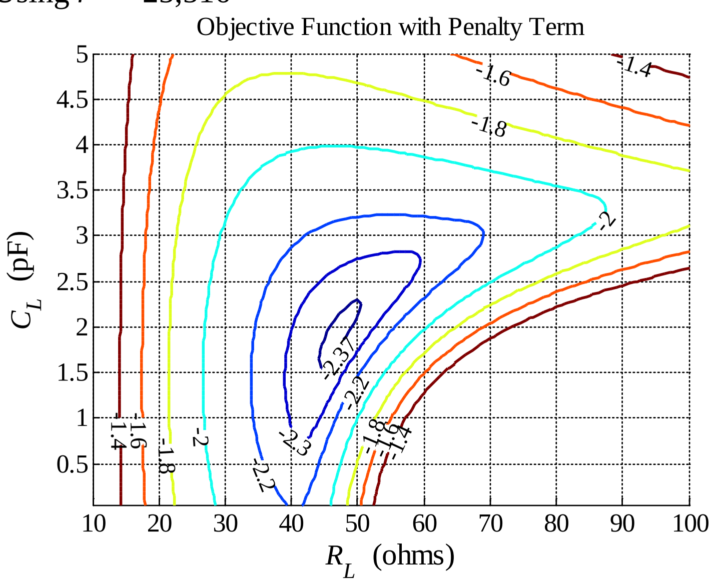


Zooming in the graphical solution it is seen that $\boldsymbol{x}^* \approx [46.1\Omega \quad 1.97\text{pF}]^T$.

b) Let us assume that the starting point is $\boldsymbol{x}_0 \approx [70\Omega \quad 1\text{pF}]^T$ (an exterior point). Then $u(\boldsymbol{x}_0) = -2.2546$, $g(\boldsymbol{x}_0) = 0.0094$. Recalculating the initial r^g,

$$r_0^g = \frac{|u(\boldsymbol{x}_0)|}{\|\boldsymbol{g}(\boldsymbol{x}_0)\|_2^2} = \frac{2.2546}{(0.0094)^2} = 25{,}516\,.$$

Using r^g = 25,516



Zooming in the graphical solution it is seen that $\boldsymbol{x}^* \approx [46.1\Omega \quad 1.97\text{pF}]^T$.