General Concepts on Numerical Optimization

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Outline

- Unconstrained optimization (terminology)
- Types of minimizers
- Examples of objective functions
- Constrained optimization (mathematical formulations)
- Linear and non-linear programming

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Unconstrained Optimization Problem

$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} u(\boldsymbol{x})$$

where u: $\Re^n \rightarrow \Re$ is the objective function (or cost function), and $\boldsymbol{x}$ contains the optimization variables

$\boldsymbol{x}^*$ is a global minimizer $\Leftrightarrow$ $u(\boldsymbol{x}^*) \leq u(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Re^n$

$\boldsymbol{x}^*$ is a local minimizer $\Leftrightarrow$ $u(\boldsymbol{x}^*) \leq u(\boldsymbol{x})$ for all $\boldsymbol{x} \in \aleph$

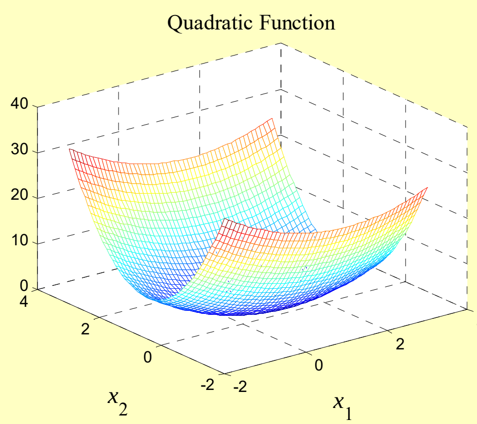
$$\aleph = \{\boldsymbol{x} \in \Re^n : \|\boldsymbol{x} - \boldsymbol{x}^*\|_2 < \varepsilon\}$$

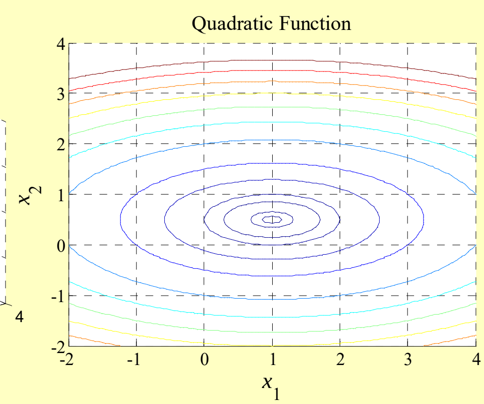
Replacing $\leq$ by $<$ we have a strict global minimizer and a strict local minimizer, respectively

If $u(\boldsymbol{x})$ is convex, a local minimizer is also the global minimizer

Example 1 – Quadratic

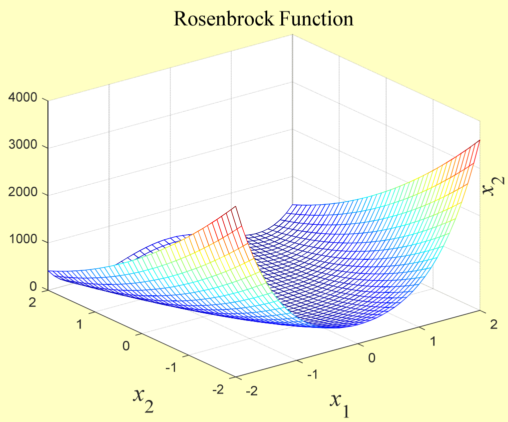
$$u(\boldsymbol{x}) = (x_1 - 1)^2 + (2x_2 - 1)^2 \quad \text{where} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

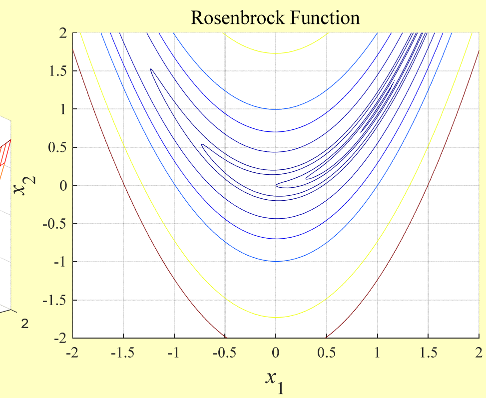




Example 2 – Rosenbrock

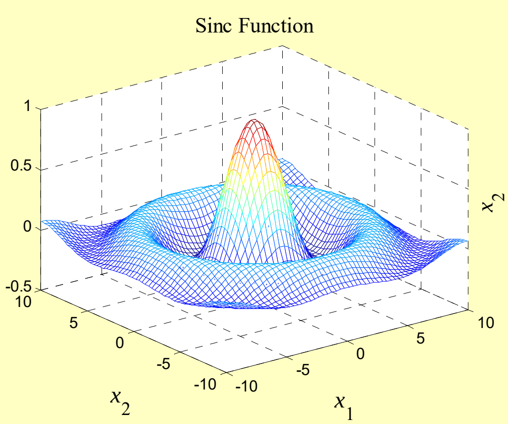
$$u(\boldsymbol{x}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \qquad \text{where} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

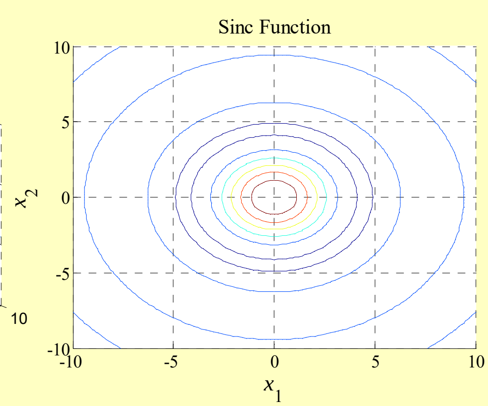




Example 3 – Sinc

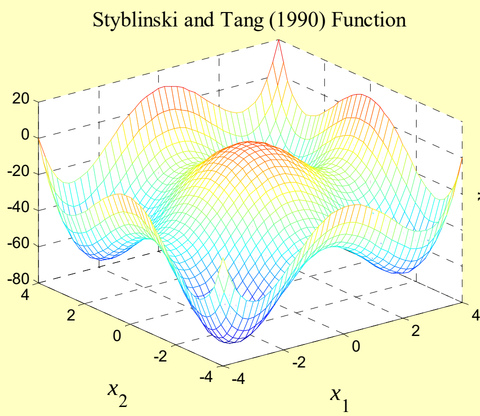
$$u(\boldsymbol{x}) = \frac{\sin(r)}{r} \qquad \text{where} \qquad r = \sqrt{x_1^2 + x_2^2} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

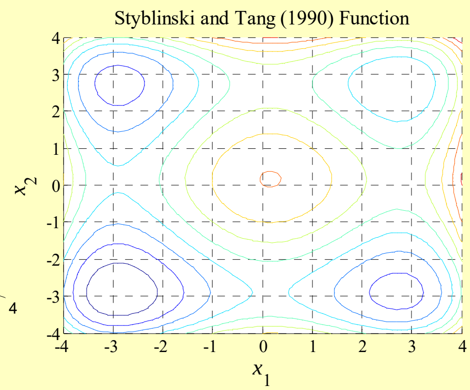




Example 4 – Styblinski and Tang

$$u(\boldsymbol{x}) = 0.5(x_1^4 - 16x_1^2 + 5x_1 + x_2^4 - 16x_2^2 + 5x_2) \quad \text{where} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$





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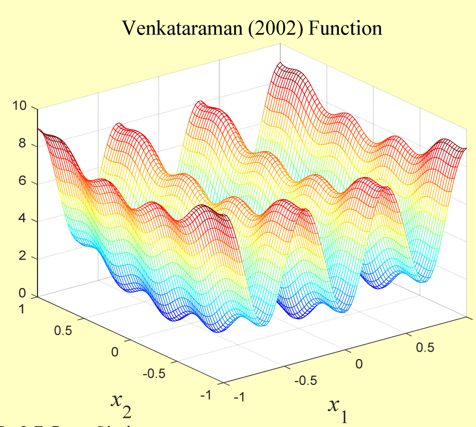
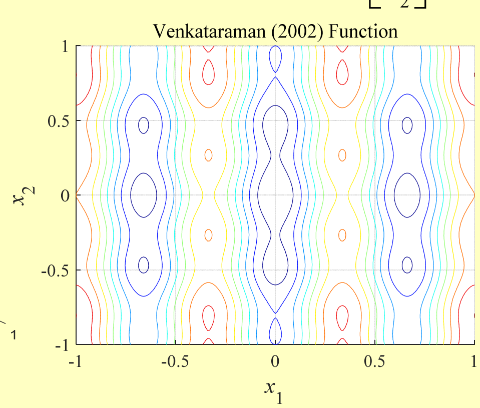
Example 5 – Venkataraman

$$u(\boldsymbol{x}) = ax_1^2 + bx_2^2 - c\cos(px_1) - d\cos(qx_2) + c + d$$

$$\text{where} \quad a = 1, b = 2, c = 0.3, d = 0.4, p = 3\pi, q = 4\pi \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$





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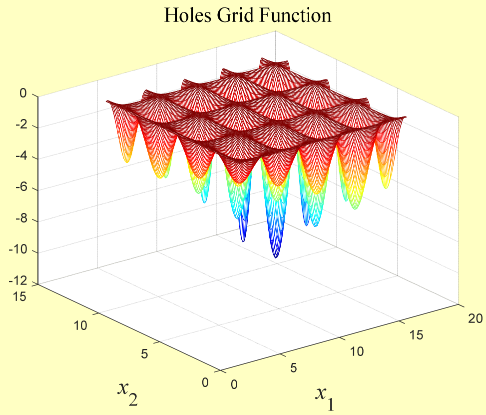
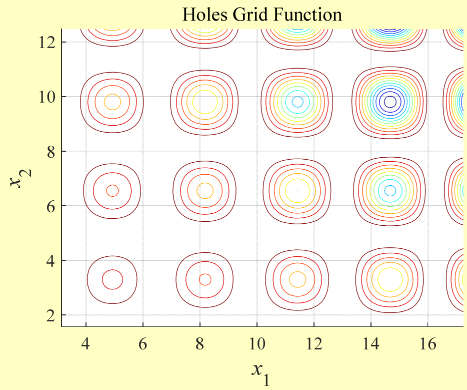
Example 6 – Holes Grid Function

$$u(\boldsymbol{x}) = -\sin^2\left(\frac{a\pi x_1}{b}\right)\cos^2\left(\frac{a\pi x_2}{b}\right)e^{\left(\frac{x_1+x_2}{c}\right)} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where a = 4, b = 13, and c = 10





Constrained Optimization Problem

Generic notation:

$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}\in\Omega} u(\boldsymbol{x})$$

u: $\Re^n \rightarrow \Re$ is the objective function (or cost function), $\boldsymbol{x}$ contains the optimization variables, Ω is the feasible set or feasible region

$\boldsymbol{x}^*$ is a global minimizer $\Leftrightarrow$ $u(\boldsymbol{x}^*) \leq u(\boldsymbol{x}),\ \boldsymbol{x} \in \Omega$

$\boldsymbol{x}^*$ is a local minimizer $\Leftrightarrow$ $u(\boldsymbol{x}^*) \leq u(\boldsymbol{x})$ for all $\boldsymbol{x} \in \aleph$

$$\aleph = \{\boldsymbol{x} \in \Omega : \|\boldsymbol{x} - \boldsymbol{x}^*\|_2 < \varepsilon\}$$

Constrained Optimization Problem (cont)

More explicit notation:

$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} u(\boldsymbol{x}) \quad \text{subject to} \begin{cases} \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0} \\ \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0} \\ \boldsymbol{x}^{\text{lb}} \leq \boldsymbol{x} \leq \boldsymbol{x}^{\text{ub}} \end{cases}$$

$\boldsymbol{h}: \Re^n \rightarrow \Re^E$ has E equality constraints, and

$\boldsymbol{g}: \Re^n \rightarrow \Re^I$ has I inequality constraints

$\boldsymbol{x}^{\text{lb}} \in \Re^n$ contains the lower bounds for $\boldsymbol{x}$, and

$\boldsymbol{x}^{\text{ub}} \in \Re^n$ the upper bounds

The last inequalities are known as side or box constraints

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Constrained Optimization Problem (cont)

Other form:

$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} u(\boldsymbol{x})$$

subject to

$$\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}$$

$$\boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}$$

$$\boldsymbol{x}^{\text{lb}} \leq \boldsymbol{x} \leq \boldsymbol{x}^{\text{ub}}$$

- Equality constraints ($\boldsymbol{h}$) are rarely used in practical problems
- In general, the number of equality constraints (E) should be smaller than the number of optimization variables (n)
- Side constraints define an hypercube (the search region)

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Linear and Nonlinear Optimization Problems

- If $u(\boldsymbol{x})$ and all the constraints are linear functions of $\boldsymbol{x}$, the problem is a linear programming problem
- If the objective function or at least one of the constraints is nonlinear, then the problem is a nonlinear programming problem
- Linear programming problems are very common in decision disciplines (marketing, sales, production planning, transportation, operational research)
- Nonlinear programming problems are very common in engineering (modeling and design)

Linear Programming Problems

Standard form:

$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} \boldsymbol{c}^T \boldsymbol{x}$$

subject to

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

$$\boldsymbol{x} \geq \boldsymbol{0}$$

- $\boldsymbol{x}, \boldsymbol{c} \in \Re^n$, $\boldsymbol{A} \in \Re^{m \times n}$, $\boldsymbol{b} \in \Re^m$
- It is assumed that $\boldsymbol{b}$ is non-negative ($\boldsymbol{b} \geq \boldsymbol{0}$, element wise), and that the rank of $\boldsymbol{A}$ is m

Non-Linear Programming Problems

Standard form:

$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} u(\boldsymbol{x})$$

subject to

$$\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}$$

$$\boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}$$

$$\boldsymbol{x}^{\mathrm{lb}} \leq \boldsymbol{x} \leq \boldsymbol{x}^{\mathrm{ub}}$$

- $\boldsymbol{x}, \boldsymbol{x}^{\mathrm{lb}}, \boldsymbol{x}^{\mathrm{ub}} \in \Re^n$
- $u: \Re^n \rightarrow \Re$, $\boldsymbol{h}: \Re^n \rightarrow \Re^E$, $\boldsymbol{g}: \Re^n \rightarrow \Re^I$
- It is assumed $n > E$