GENERAL CONCEPTS ON NUMERICAL OPTIMIZATION

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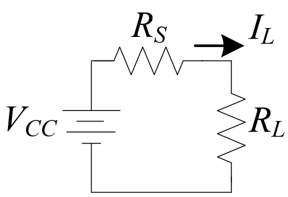
EXAMPLES OF GRAPHICAL SOLUTIONS OF NON-LINEAR PROGRAMMING PROBLEMS

1. Maximizing Power Transfer in a Simple DC Circuit

Consider the following simple DC circuit. Assuming that V_{CC} = 12 V, we want to find the optimal values of R_S and R_L that maximize the power delivered to the load, keeping the load current at I_L = 10 mA. The minimum and maximum values allowed for R_S and R_L are 200 Ω and 1 KΩ, respectively.



The standard formulation as a nonlinear programming problem is:

$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} u(\boldsymbol{x})$$

subject to

$$\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}$$

$$\boldsymbol{g}(\boldsymbol{x}) \le \boldsymbol{0}$$

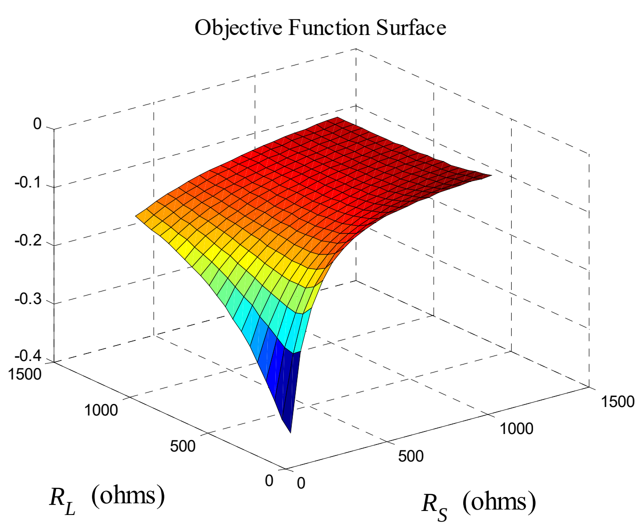
$$\boldsymbol{x}^{\text{lb}} \le \boldsymbol{x} \le \boldsymbol{x}^{\text{ub}}$$

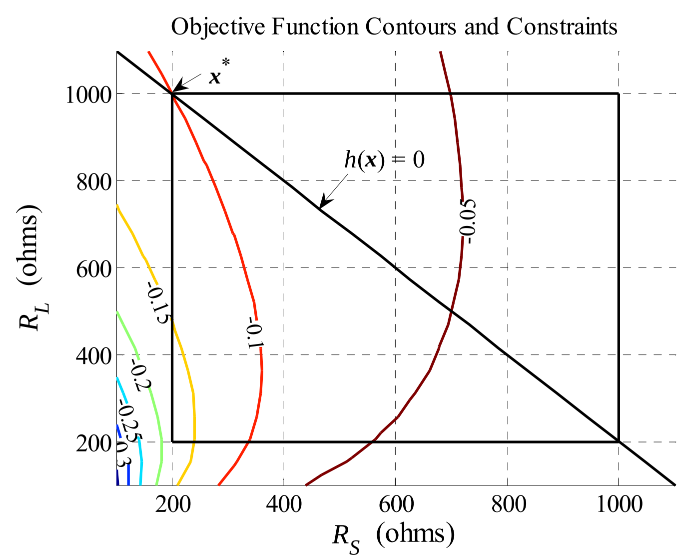
where $\boldsymbol{x}$, $\boldsymbol{x}_{\text{lb}}$, $\boldsymbol{x}_{\text{ub}} \in \Re^n$, u: $\Re^n \rightarrow \Re$, $\boldsymbol{h}$: $\Re^n \rightarrow \Re^E$, $\boldsymbol{g}$: $\Re^n \rightarrow \Re^I$.

In this example the optimization variables are $\boldsymbol{x} = [R_S \quad R_L]^{\text{T}}$, and then n = 2. Since we want to maximize the power at the load, $u(\boldsymbol{x}) = -P_L$, where $P_L = \left(\frac{V_{CC}}{R_S + R_L}\right)^2 R_L$. Then $u(\boldsymbol{x}) = -\left(\frac{12}{x_1 + x_2}\right)^2 x_2$.

Since $I_L = 10$ mA and $I_L = \frac{V_{CC}}{R_S + R_L}$, then $h(\boldsymbol{x}) = \frac{12}{x_1 + x_2} - 0.01$.

Finally, $\boldsymbol{x}^{\text{lb}} = [200 \quad 200]^{\text{T}}$ and $\boldsymbol{x}^{\text{ub}} = [1000 \quad 1000]^{\text{T}}$. It is seen that E = 1, and I = 0.





The optimal solution is $\boldsymbol{x}^* = [200 \quad 1000]^{\text{T}}$. The optimal response is $P_L^* = P_L(\boldsymbol{x}^*) =$ 0.1 W.

2. Paper Sheet Function

Assume we want to minimize a paper sheet function given by $y = (x_1 - 1)^2 + x_2 - 2$, subject to $x_2 - x_1 = 1$ and $x_1 + x_2 \le 2$. Considering the following standard formulation of a nonlinear programming problem:

$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} u(\boldsymbol{x})$$

$$\text{subject to}$$

$$\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}$$

$$\boldsymbol{g}(\boldsymbol{x}) \le \boldsymbol{0}$$

$$\boldsymbol{x}^{\text{lb}} \le \boldsymbol{x} \le \boldsymbol{x}^{\text{ub}}$$

where $\boldsymbol{x}, \boldsymbol{x}_{\text{lb}}, \boldsymbol{x}_{\text{ub}} \in \Re^n$, $u: \Re^n \to \Re$, $\boldsymbol{h}: \Re^n \to \Re^E$, $\boldsymbol{g}: \Re^n \to \Re^I$

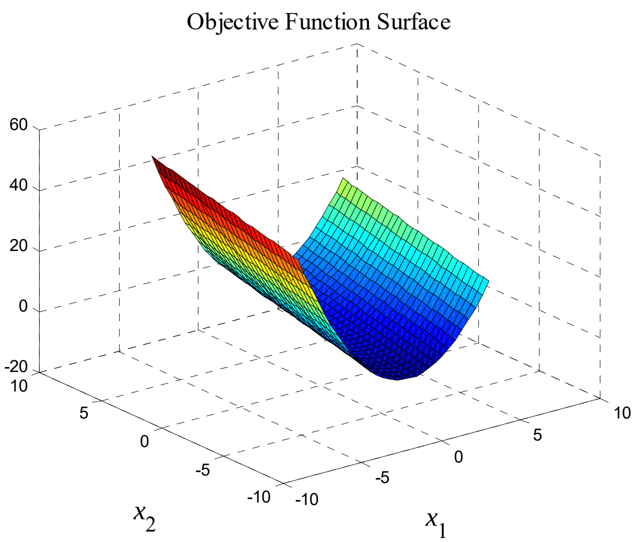
Hence,

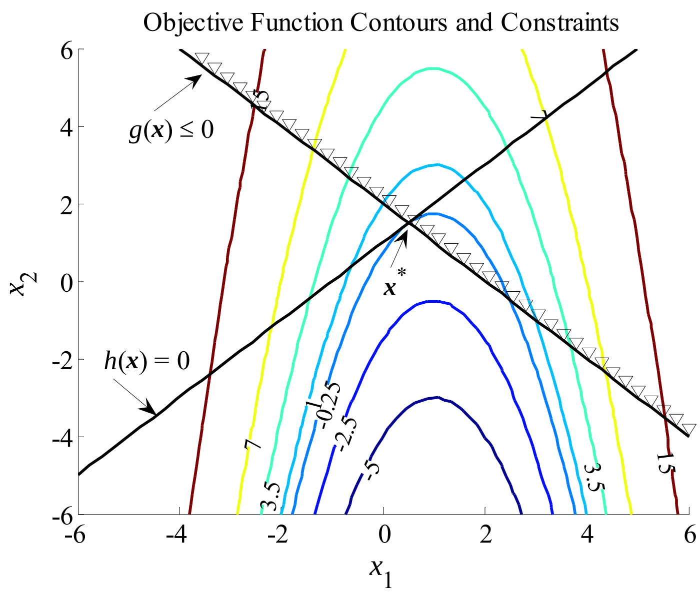
$$u(\boldsymbol{x}) = (x_1 - 1)^2 + x_2 - 2$$

$$h(\boldsymbol{x}) = x_2 - x_1 - 1$$

$$g(\boldsymbol{x}) = x_1 + x_2 - 2$$

$n = 2$, $E = 1$, and $I = 1$, with no box constraints.



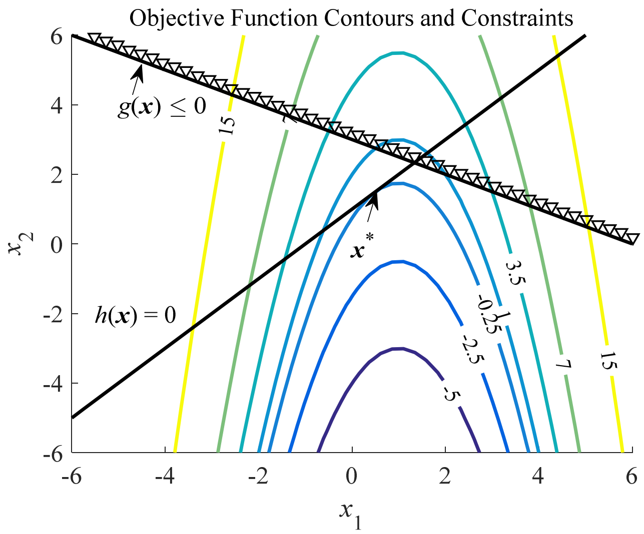


The optimal solution is $\boldsymbol{x}^* = [0.5 \quad 1.5]^{\text{T}}$. The optimal response is $u(\boldsymbol{x}^*) = -0.25$. It is seen that the inequality constraint is inactive.

Notice that, in this example, $\nabla u(\boldsymbol{x}^*)$ and $\nabla h(\boldsymbol{x}^*)$ are parallel vectors (the contour plot and the equality plot are tangential at the solution). This situation is known as the Lagrange condition. This condition is no longer valid if the inequality constraint becomes active, as shown next.

Using other inequality constraints:

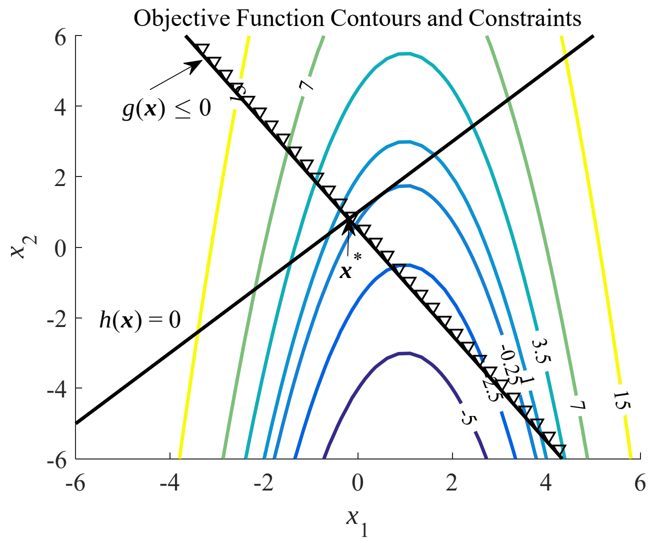
$$g(\boldsymbol{x}) = 0.5x_1 + x_2 - 3$$



$$\boldsymbol{x}^* = [0.5 \quad 1.5]^{\mathrm{T}}\ ,\ u(\boldsymbol{x}^*) = -0.25$$

Lagrange condition is again satisfied (inequality constraint is inactive)

$$g(\boldsymbol{x}) = 1.5x_1 + x_2 - 0.5$$



$$\boldsymbol{x}^* = [-0.2 \quad 0.8]^{\mathrm{T}}\ ,\ u(\boldsymbol{x}^*) = 0.24$$

Lagrange condition is not satisfied in this case (inequality constraint is active)