CONSTRAINED OPTIMIZATION

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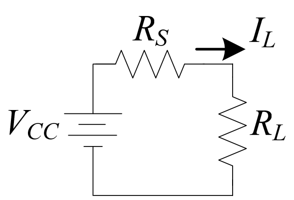
March 6, 2013

GRAPHICAL EXAMPLES OF CONSTRAINED NON-LINEAR PROGRAMMING PROBLEMS

1. Maximizing Power Transfer in a Simple DC Circuit

Consider the following simple DC circuit. Assuming that V_{CC} = 12 V, we want to find the optimal values of R_S and R_L that maximize the power delivered to the load, keeping the load current at I_L = 10 mA. The minimum and maximum values allowed for R_S and R_L are 200Ω and 1KΩ, respectively.



The standard formulation as a nonlinear programming problem is:

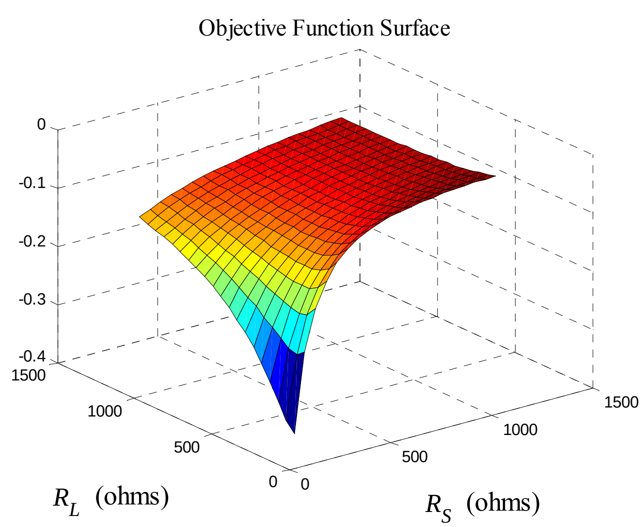
$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} u(\boldsymbol{x})$$
$$\text{subject to}$$
$$\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}$$
$$\boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}$$
$$\boldsymbol{x}^{\text{lb}} \leq \boldsymbol{x} \leq \boldsymbol{x}^{\text{ub}}$$

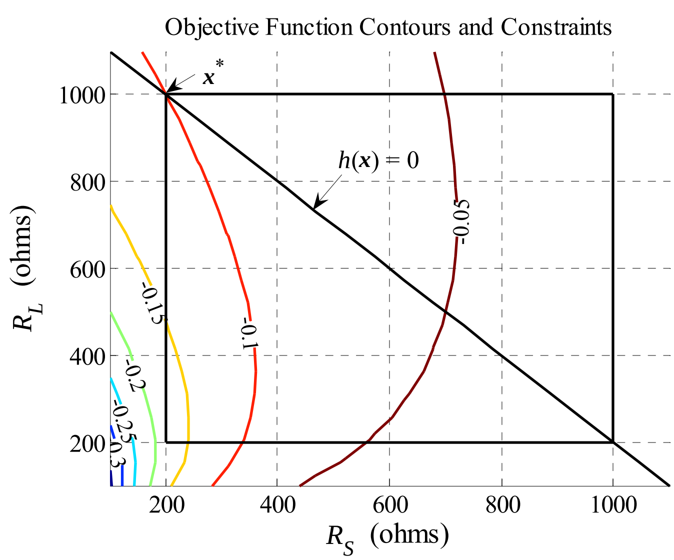
where $\boldsymbol{x}, \boldsymbol{x}_{\text{lb}}, \boldsymbol{x}_{\text{ub}} \in \Re^n$, u: $\Re^n \rightarrow \Re$, $\boldsymbol{h}$: $\Re^n \rightarrow \Re^E$, $\boldsymbol{g}$: $\Re^n \rightarrow \Re^I$.

In this example the optimization variables are $\boldsymbol{x} = [R_S \quad R_L]^{\text{T}}$. Since we want to maximize the power at the load, $u(\boldsymbol{x}) = -P_L$, where $P_L = \left(\frac{V_{CC}}{R_S + R_L}\right)^2 R_L$. Then $u(\boldsymbol{x}) = -\left(\frac{12}{x_1 + x_2}\right)^2 x_2$.

Since $I_L = 10\text{ mA}$ and $I_L = \frac{V_{CC}}{R_S + R_L}$, then $h(\boldsymbol{x}) = \frac{12}{x_1 + x_2} - 0.01$.

Finally, $\boldsymbol{x}^{\text{lb}} = [200 \quad 200]^{\text{T}}$ and $\boldsymbol{x}^{\text{ub}} = [1000 \quad 1000]^{\text{T}}$. It is seen that n = 2, E = 1, and I = 0.





The optimal solution is $\boldsymbol{x}^* = [200 \quad 1000]^{\text{T}}$. The optimal response is $P_L^* = P_L(\boldsymbol{x}^*) = 0.1$ W.

The previous problem can be formulated as an indirect unconstrained problem:

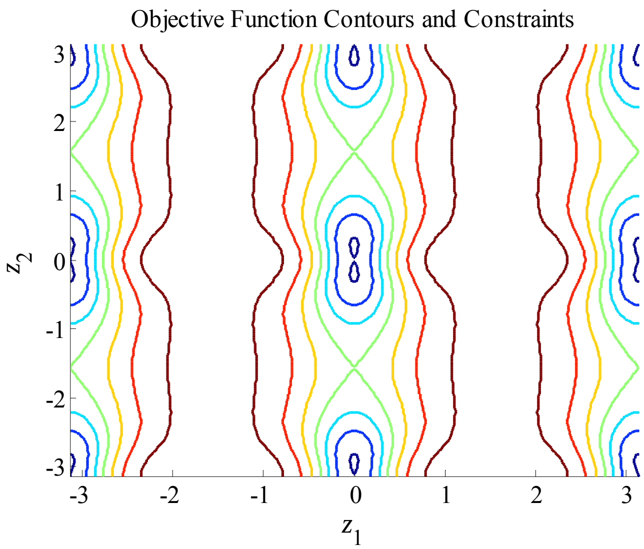
$\boldsymbol{z}^* = \arg\min_{\boldsymbol{z}} U(\boldsymbol{z})$, where

$$U(\boldsymbol{z}) = u(\boldsymbol{x}) + r_{\text{h}}(h(\boldsymbol{x}))^2 \; ; \;\; u(\boldsymbol{x}) = -\left(\frac{12}{x_1 + x_2}\right)^2 x_2 \;\; ; \;\; h(\boldsymbol{x}) = \frac{12}{x_1 + x_2} - 0.01$$

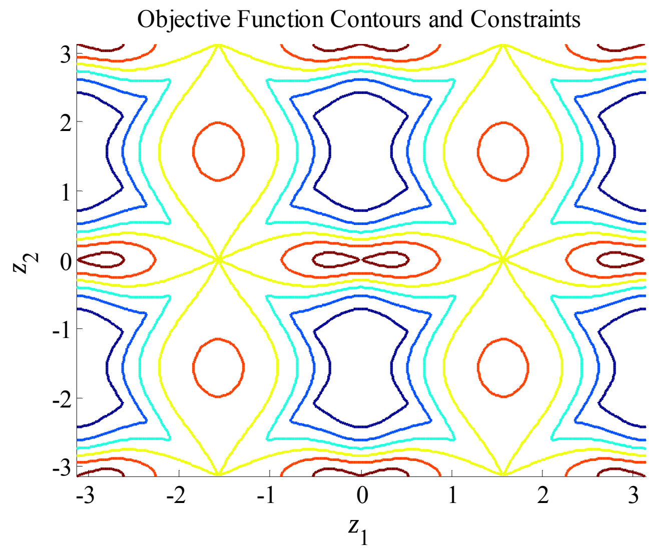
and $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^{\text{lb}} + (x_1^{\text{ub}} - x_1^{\text{lb}})(\sin z_1)^2 \\ x_2^{\text{lb}} + (x_2^{\text{ub}} - x_2^{\text{lb}})(\sin z_2)^2 \end{bmatrix}$ with $\boldsymbol{x}^{\text{lb}} = [200 \quad 200]^{\text{T}}$ and $\boldsymbol{x}^{\text{ub}} = [1000 \quad 1000]^{\text{T}}$.

The optimal solution found, $\boldsymbol{z}^*$, depends on the value of the penalty term r_{h}.

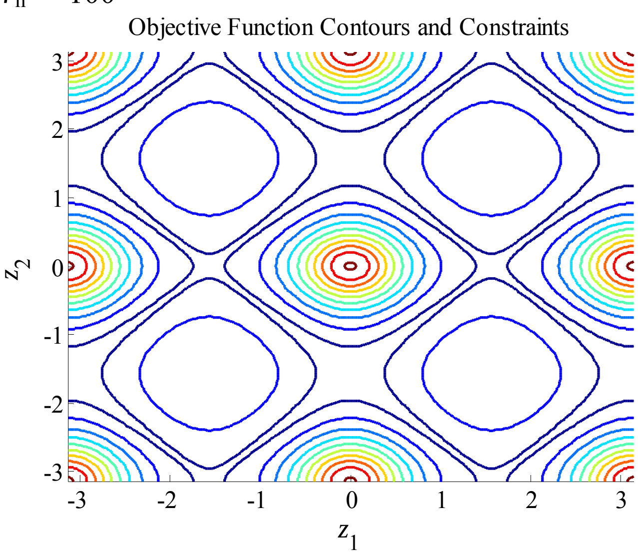
$r_{\text{h}} = 1$



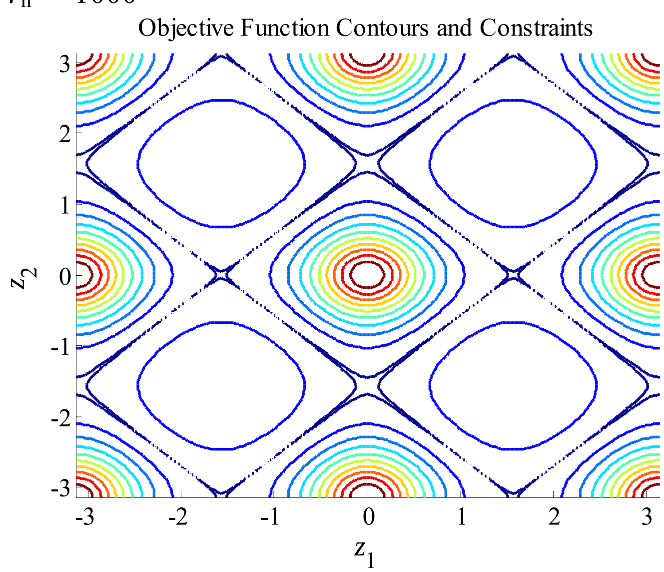
$r_{\text{h}} = 10$



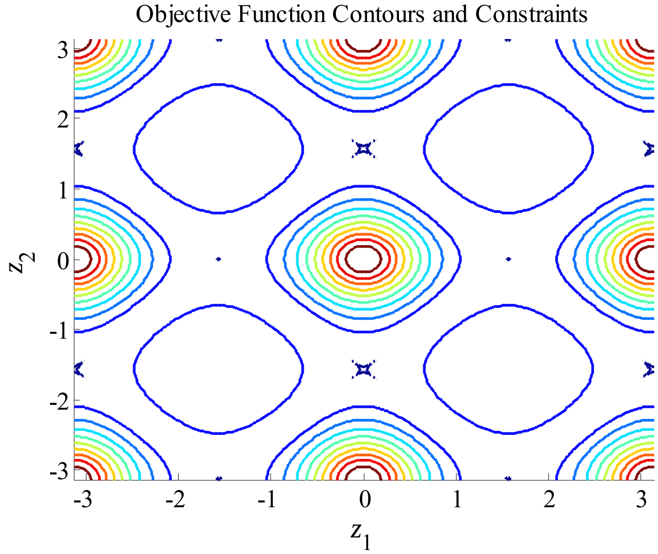
$r_{\text{h}} = 100$



$r_{\text{h}} = 1000$



$r_{\mathrm{h}} = 10{,}00$



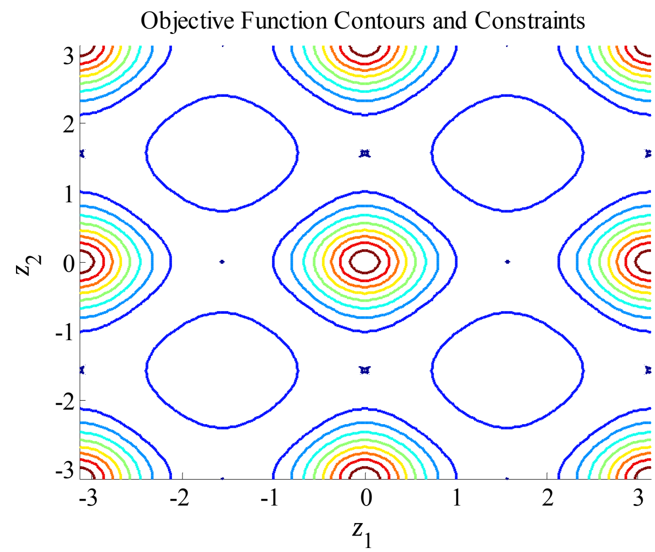
It is seen that the problem can be solved in successive unconstrained optimizations, increasing the value of r_{h} geometrically at each optimization.

It is also seen that all the local minima of the transformed problem correspond to the same solution in the original problem:

$$z_1^* = -\pi, 0, \pi, \ldots \text{ and } z_2^* = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

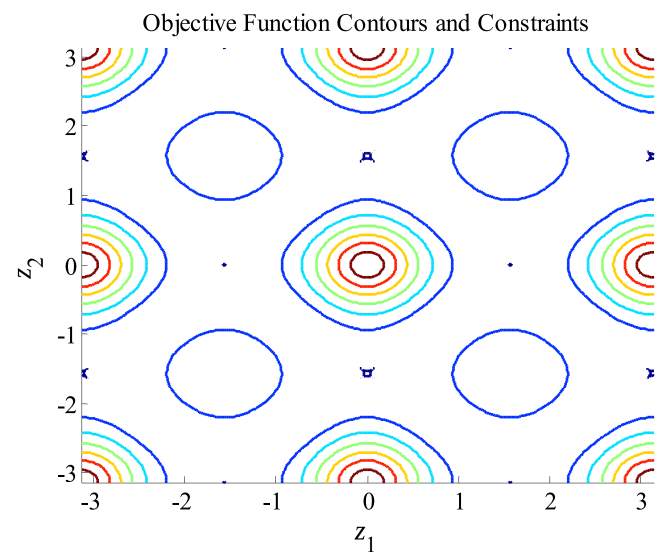
If the starting point is $\boldsymbol{x}_0 = [800 \quad 600]^T$, then $u(\boldsymbol{x}_0) = -\,0.0441$, $h(\boldsymbol{x}_0) = -0.0014$. A better way to choose the initial r^{h} is

$$r_0^{\mathrm{h}} = \frac{|u(\boldsymbol{x}_0)|}{\|\boldsymbol{h}(\boldsymbol{x}_0)\|_2^2} = \frac{0.0441}{(0.0014)^2} = 21{,}600$$



If the starting point is $\boldsymbol{x}_0 = [20 \quad 900]^T$, then $u(\boldsymbol{x}_0) = -\,0.1531$, $h(\boldsymbol{x}_0) = -0.003$. A better way to choose the initial r^{h} is

$$r_0^{\mathrm{h}} = \frac{|u(\boldsymbol{x}_0)|}{\|\boldsymbol{h}(\boldsymbol{x}_0)\|_2^2} = \frac{0.1531}{(0.003)^2} = 16{,}531$$



2. Paper Sheet Function

Assume we want to minimize a paper sheet function given by $y = (x_1 - 1)^2 + x_2 - 2$, subject to $x_2 - x_1 = 1$ and $x_1 + x_2 \le 2$. Considering the following standard formulation of a nonlinear programming problem:

$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} u(\boldsymbol{x})$$

subject to

$$\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}$$

$$\boldsymbol{g}(\boldsymbol{x}) \le \boldsymbol{0}$$

$$\boldsymbol{x}^{\text{lb}} \le \boldsymbol{x} \le \boldsymbol{x}^{\text{ub}}$$

where $\boldsymbol{x}$, $\boldsymbol{x}_{\text{lb}}$, $\boldsymbol{x}_{\text{ub}} \in \Re^n$, u: $\Re^n \rightarrow \Re$, $\boldsymbol{h}$: $\Re^n \rightarrow \Re^E$, $\boldsymbol{g}$: $\Re^n \rightarrow \Re^I$

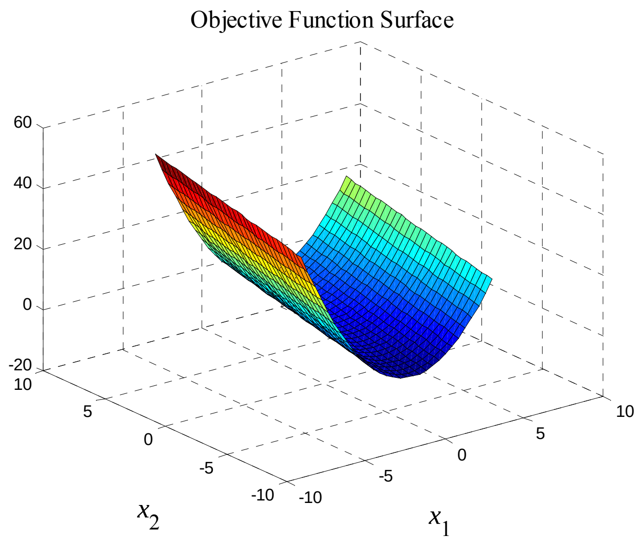
Hence,

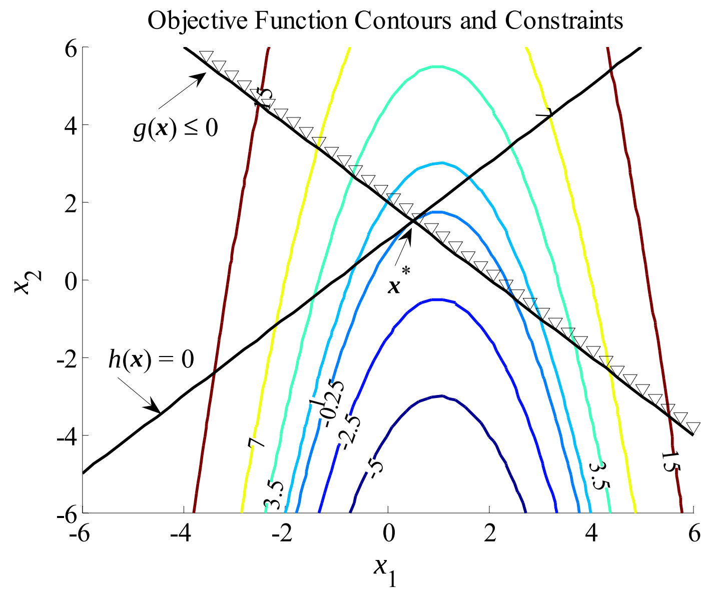
$$u(\boldsymbol{x}) = (x_1 - 1)^2 + x_2 - 2$$

$$h(\boldsymbol{x}) = x_2 - x_1 - 1$$

$$g(\boldsymbol{x}) = x_1 + x_2 - 2$$

$n = 2$, $E = 1$, and $I = 1$, with no box constraints.





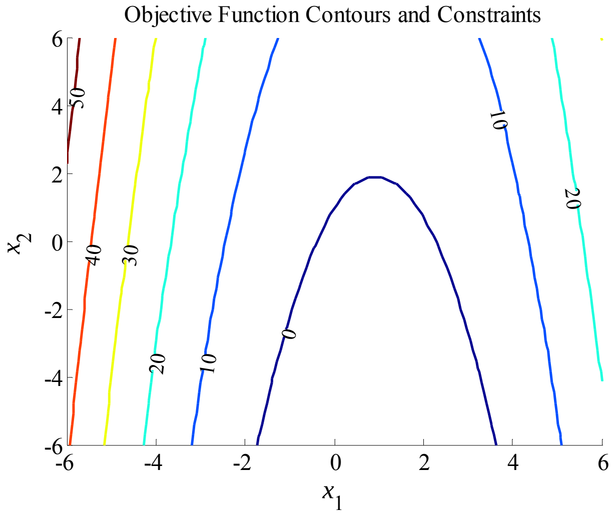
The optimal solution is $\boldsymbol{x}^* = [0.5 \quad 1.5]^{\text{T}}$. It is seen that the inequality constraint $g(\boldsymbol{x})$ does not affect the optimal solution $\boldsymbol{x}^*$ due to the form of the objective function $u(\boldsymbol{x})$.

This problem can also be formulated as an indirect unconstrained optimization problem,

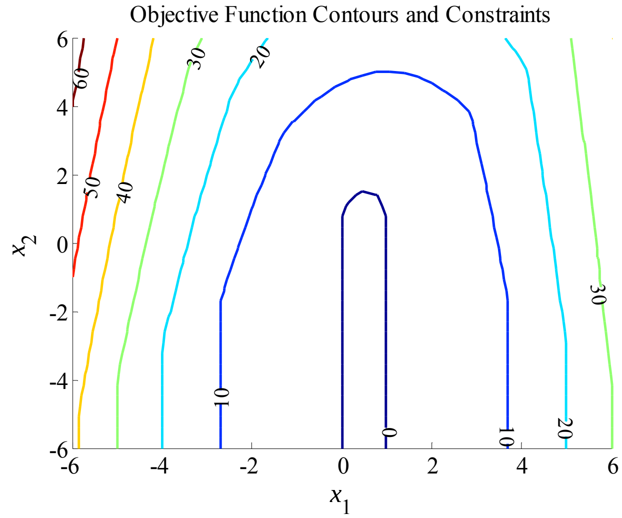
$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} U(\boldsymbol{x}) \text{ where } U(\boldsymbol{x}) = u(\boldsymbol{x}) + r_{\text{h}}(h(\boldsymbol{x}))^2 + r_{\text{g}}(\max\{0, g(\boldsymbol{x})\})^2$$

The optimal solution found, $\boldsymbol{x}^*$, depends on the values of the penalty terms r_{h} and r_{g}.

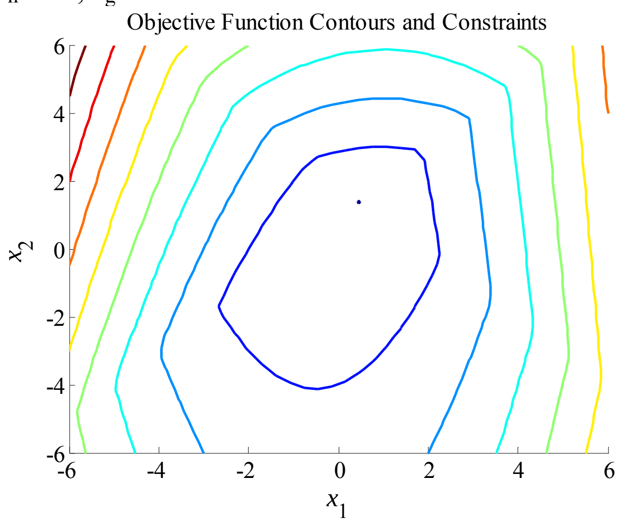
$r_h = 0.1, r_g = 0.1$



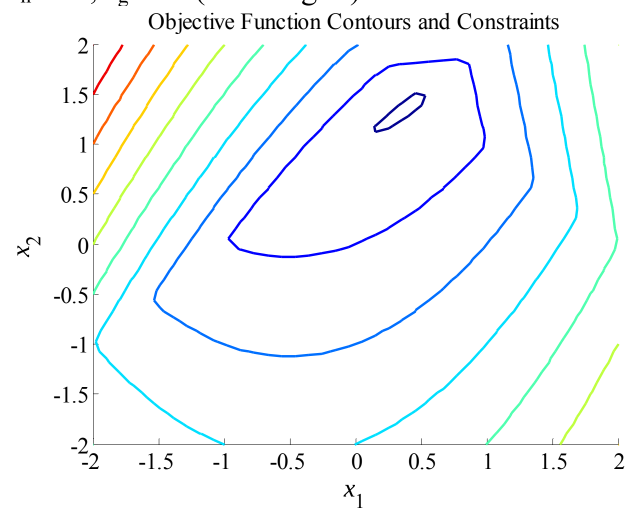
$r_h = 1, r_g = 1$



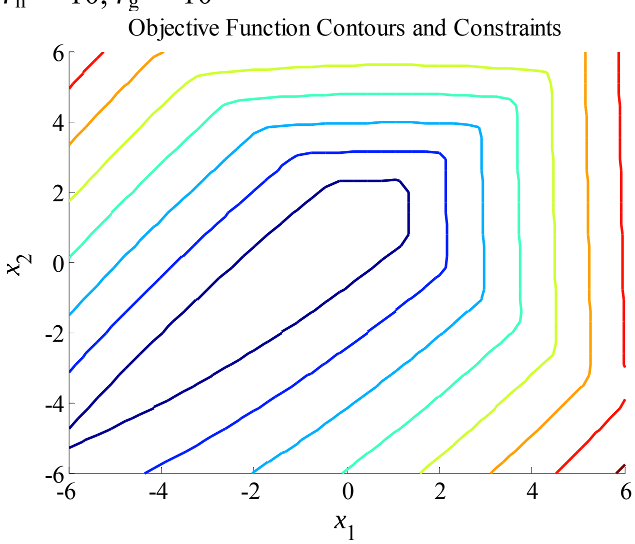
$r_h = 3, r_g = 3$



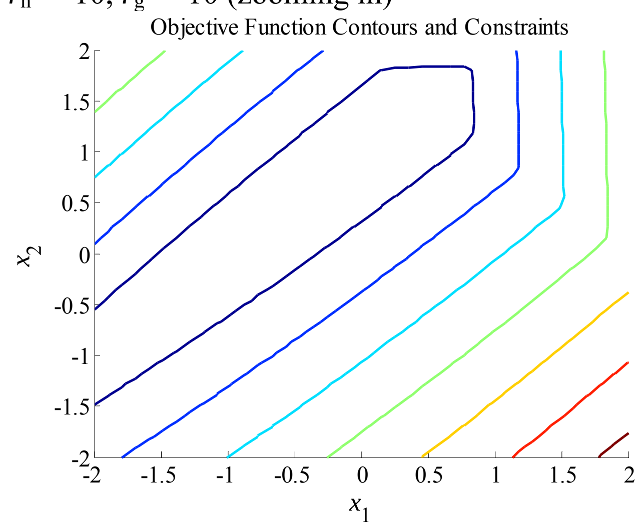
$r_h = 3, r_g = 3$ (zooming in)



$r_h = 10, r_g = 10$

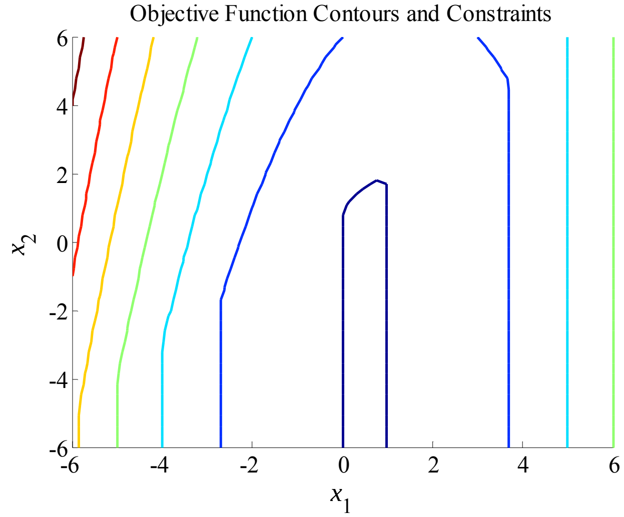


$r_h = 10, r_g = 10$ (zooming in)

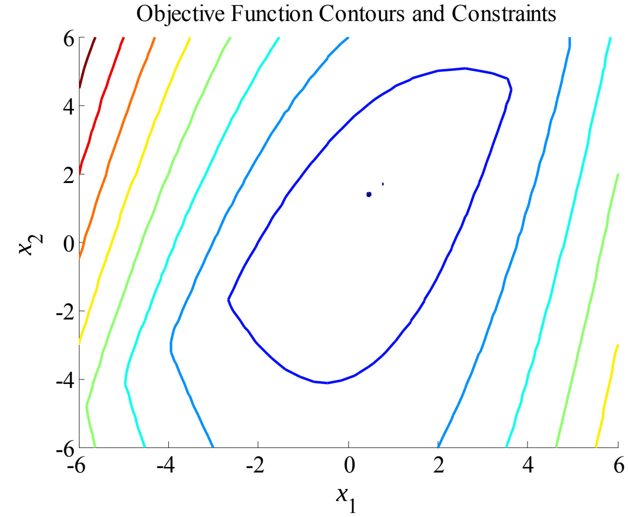


This case illustrates the problem of over-emphasizing the constraints when the selected penalty terms are too large.

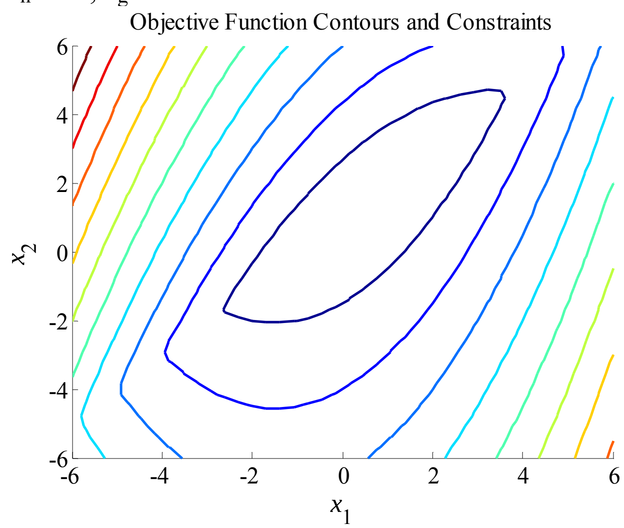
$r_h = 1$, $r_g = 0$



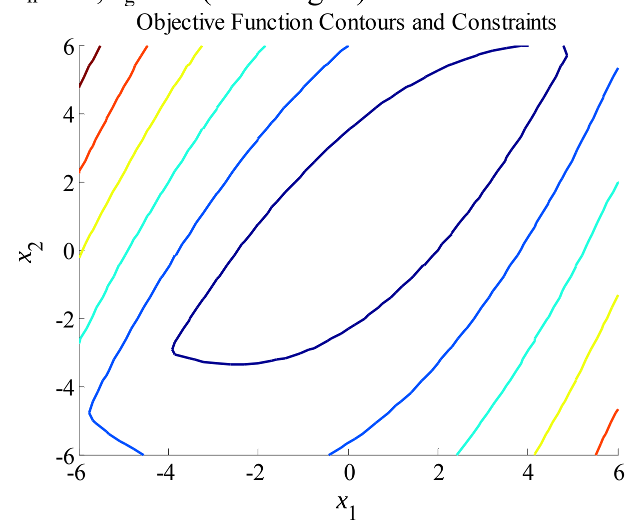
$r_h = 3$, $r_g = 0$



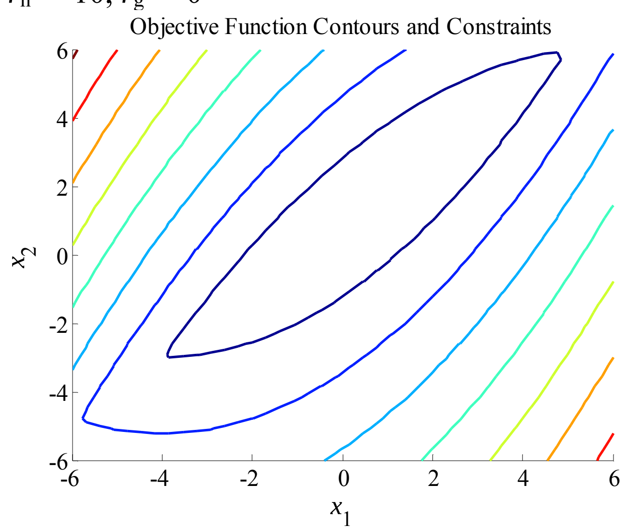
$r_h = 5$, $r_g = 0$



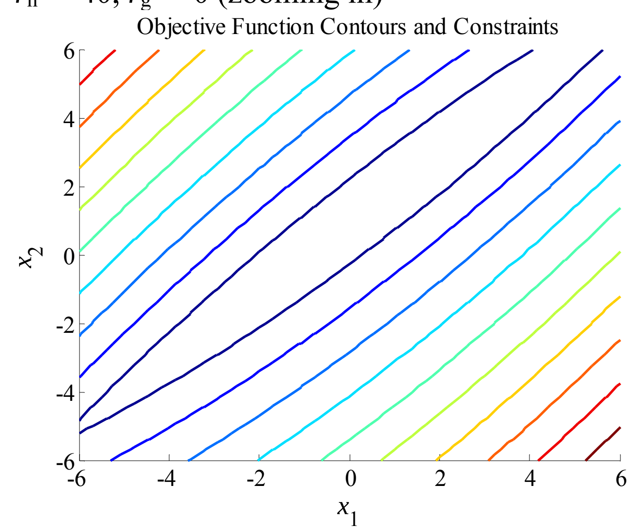
$r_h = 7$, $r_g = 0$ (zooming in)



$r_h = 10$, $r_g = 0$



$r_h = 40$, $r_g = 0$ (zooming in)



The above contours confirm that, in this particular case, the inequality constraint $g(\boldsymbol{x})$ does not affect the optimal solution $\boldsymbol{x}^*$ due to the form of the objective function $u(\boldsymbol{x})$. It is also seen that, if r_h is too large, the equality constraint $h(\boldsymbol{x})$ becomes dominant.