A Review on Matrix Computations
(Part 2)

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Outline

- Basics on multi-dimensional geometry
- Convex and non-convex functions
- Examples

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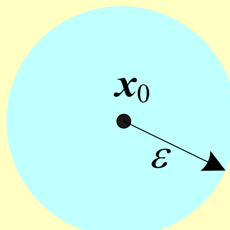
Neighborhood

The neighborhood of a point $\boldsymbol{x}_0$ is the set

$$\{\boldsymbol{x} \in \Re^n : \|\boldsymbol{x} - \boldsymbol{x}_0\|_2 < \varepsilon\}$$

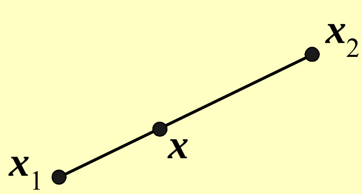
where ε is some positive number



Line Segments

A line segment L between points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is the set of points

$$L = \{\boldsymbol{x} \in \Re^n : \boldsymbol{x} = \lambda\boldsymbol{x}_2 + (1-\lambda)\boldsymbol{x}_1, \lambda \in [0,1]\}$$



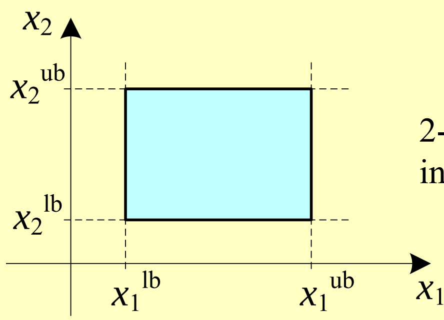
Multi-Dimensional Interval

Multidimensional intervals are denoted as

$$X = [\boldsymbol{x}^{\text{lb}}, \boldsymbol{x}^{\text{ub}}] = \{\boldsymbol{x} \in \Re^n : \boldsymbol{x}^{\text{lb}} \leq \boldsymbol{x} \leq \boldsymbol{x}^{\text{ub}}\}$$

$X \subseteq \Re^n$ is the n-dimensional interval between $\boldsymbol{x}^{\text{lb}}$ and $\boldsymbol{x}^{\text{ub}}$

where "≤" is an element-wise operator



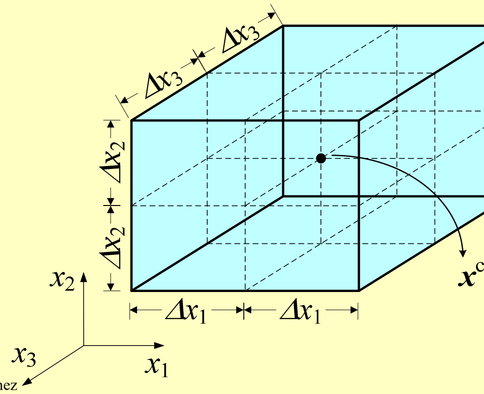
2-dimensional interval

Multi-Dimensional Interval (cont)

$X \subseteq \Re^n$ as an n-dimensional interval centered at $\boldsymbol{x}^{\text{c}}$ with size $||2\Delta\boldsymbol{x}||_2$

$$X(\boldsymbol{x}^{\text{c}}, \Delta\boldsymbol{x}) = [\boldsymbol{x}^{\text{c}} - \Delta\boldsymbol{x}, \boldsymbol{x}^{\text{c}} + \Delta\boldsymbol{x}]$$

$$X(\boldsymbol{x}^{\text{c}}, \Delta\boldsymbol{x}) = \{\boldsymbol{x} \in \Re^n : (\boldsymbol{x}^{\text{c}} - \Delta\boldsymbol{x}) \leq \boldsymbol{x} \leq (\boldsymbol{x}^{\text{c}} + \Delta\boldsymbol{x})\}$$

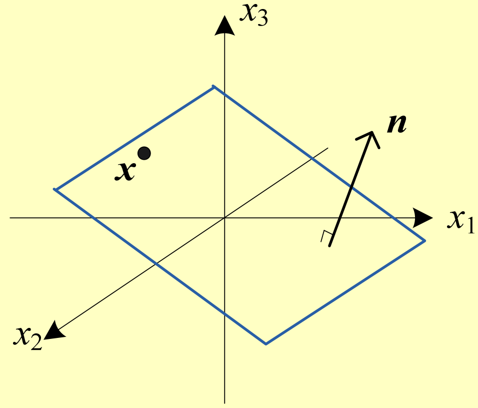


3-dimensional interval centered at $\boldsymbol{x}^{\text{c}}$ with size $||2\Delta\boldsymbol{x}||_2$

Hyperplanes

Let $\boldsymbol{n} \in \Re^n$ and $v \in \Re$. A hyperplane H is the set of points

$$H = \{\boldsymbol{x} \in \Re^n : \boldsymbol{n}^T \boldsymbol{x} = v\}$$



Hyperplanes (cont)

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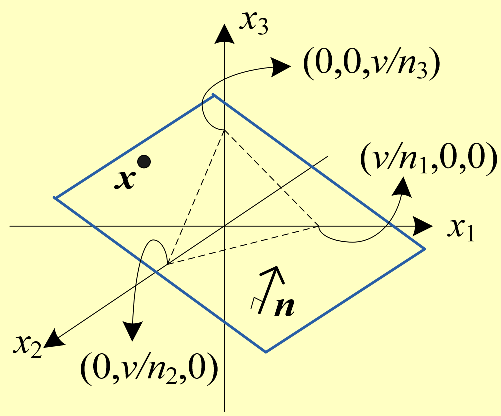
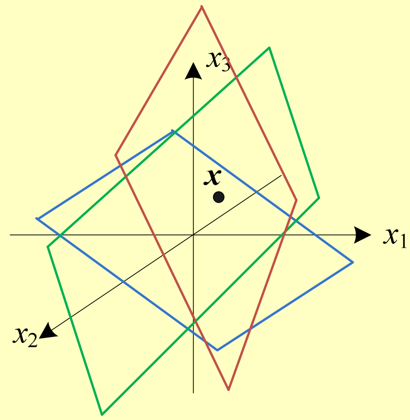


Linear Variety

Let $\boldsymbol{A} \in \Re^{m \times n}$, $\boldsymbol{x} \in \Re^{n}$, and $\boldsymbol{b} \in \Re^{m}$. A linear variety L_v is the intersection of a finite number of hyperplanes

$$L_v = \{\boldsymbol{x} \in \Re^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$$



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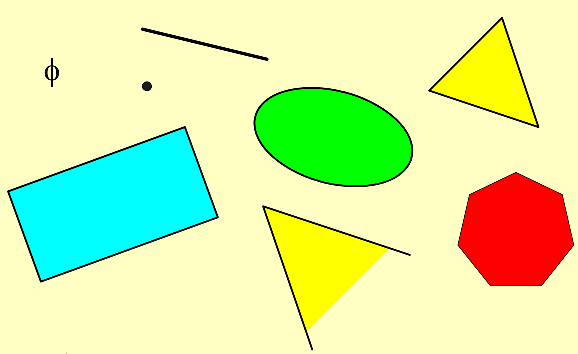
Convex Set

- A set is convex if the line segment connecting any two points in the set lies entirely in the set
- Ω is convex if for any $\boldsymbol{a}, \boldsymbol{b} \in \Omega$, $\lambda\boldsymbol{a}+(1-\lambda)\boldsymbol{b} \in \Omega$, $\lambda \in [0,1]$
- Examples of convex sets:



The intersection of two convex sets is convex

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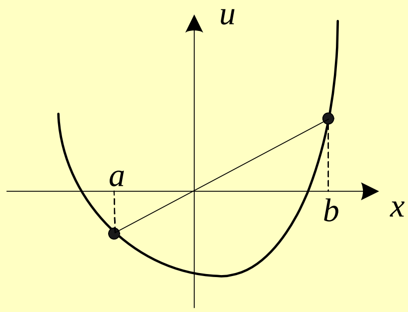
Convex Function

A multidimensional scalar function u: $\Re^n \rightarrow \Re$ is convex if for any $\boldsymbol{a}$, $\boldsymbol{b} \in \Re^n$, with $\lambda \in [0,1]$, then

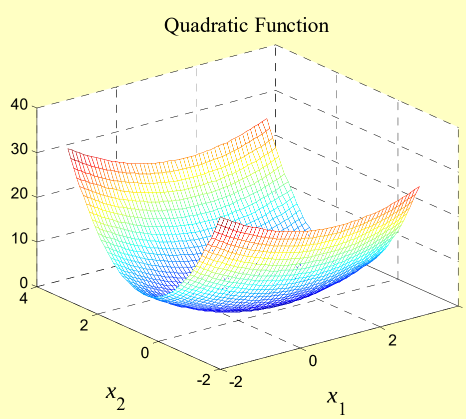
$$u(\lambda \boldsymbol{a} + (1-\lambda)\boldsymbol{b}) \leq \lambda u(\boldsymbol{a}) + (1-\lambda) u(\boldsymbol{b})$$

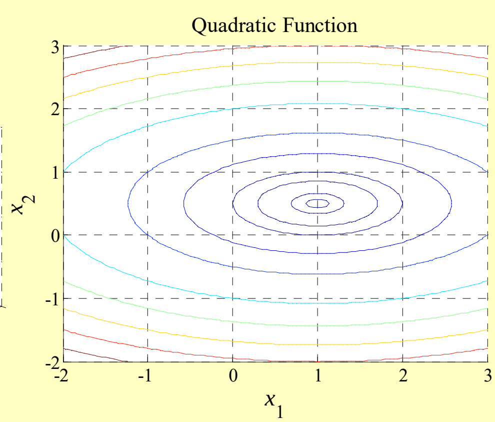


The linear interpolation between any two function values is equal to the function or overestimates the function

Example of a Convex Function

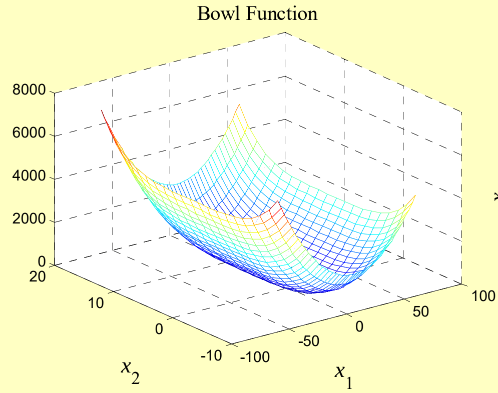
$$y(\boldsymbol{x}) = (x_1 - 1)^2 + (2x_2 - 1)^2 \qquad \text{where} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

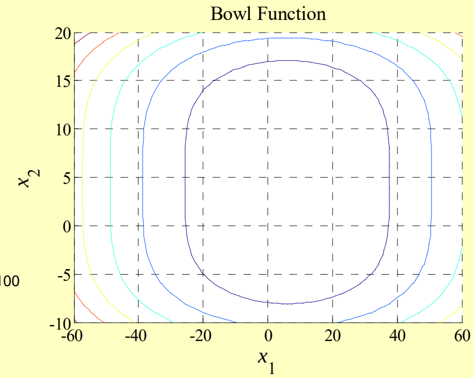




Example of a Convex Function

$$y(\boldsymbol{x}) = (x_1 - 6)^2 + \frac{1}{25}(x_2 - 4.5)^4 \quad \text{where} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$





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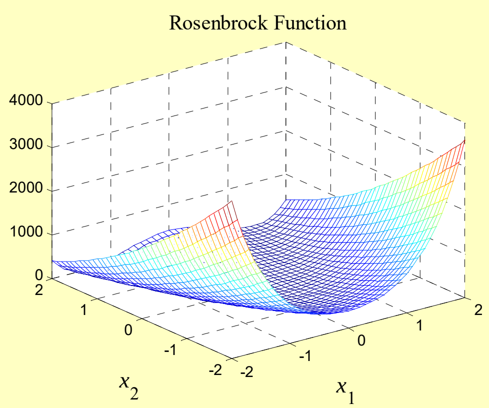
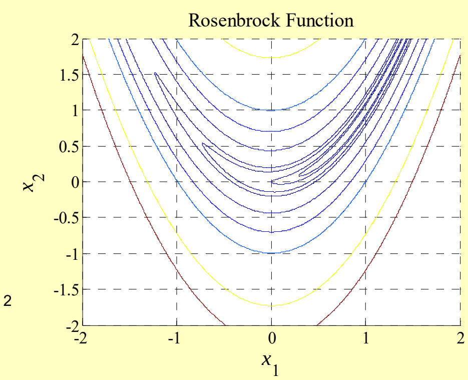
Example of a Nonconvex Function (cont)

$$y(\boldsymbol{x}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \quad \text{where} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$





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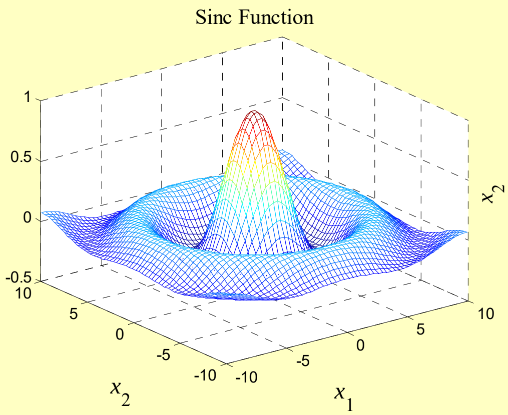
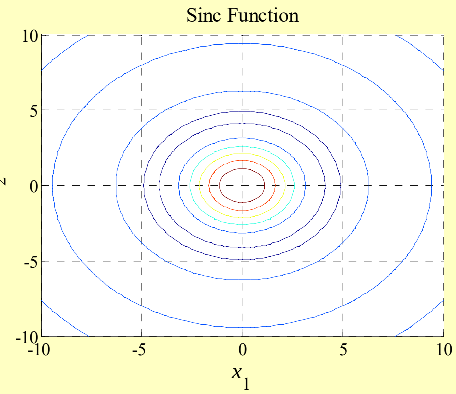
Example of a Nonconvex Function (cont)

$$y(\boldsymbol{x}) = \frac{\sin(r)}{r} \qquad \text{where} \qquad r = \sqrt{x_1^2 + x_2^2} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$





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